Proof of Theorem 10.3, Part 1 (VERSION of JULY 31, 2014, 11:05am (my time))

By DZ

Theorem 10.3 Let $\lambda \vdash n$ and $\tau \vdash n+1$.

1. Assume first that $\tau \neq (\mu, 1)$, namely $\tau = (\tau_1, \ldots, \tau_k)$ where $\tau_k \geq 2$, then

$$\sum_{\nu \in \lambda^+} \chi^\nu(\tau) = 0.$$

2. Let $\lambda, \mu \vdash n$ then

$$\sum_{\nu \in \lambda^+} \chi^\nu(\mu, 1) = (N_1(\mu) + 1) \cdot \chi^\lambda(\mu),$$

where $N_1(\mu)$ is the number of ones in μ.

Remark: A better way to state the theorem is:

$$\sum_{\nu \in \lambda^+} \chi^\nu(\tau) \; = \; N_1(\tau) \cdot \chi^\lambda(\mu) \quad ,$$

where $\mu = (\tau_1, \ldots, \tau_{k-1}, \tau_k - 1)$ (and remove the last 0 if $\tau_k = 1$). The proof is by induction on the number ones in τ, and part 1 is the base case.

Important Definition Let $F(x_1, \ldots, x_n)$ be any Laurent polynomial. Then the **constant term** of F, denoted by $CT_{x_1,\ldots,x_n} F$, or simply CTF when there is no ambiguity, is the coefficient of $x_1^0 \cdots x_n^0$ in F.

Examples:

$$CT_{x_1,x_2}(3 + x_1 + x_2^{-1}) = 3 \quad ,$$

$$CT_{x_1}(3 + x_1 + x_2^{-1}) = 3 + x_2^{-1} \quad .$$

Proof of Part 1: One way to define the characters $\chi^\lambda(\mu)$ is via the formula mentioned on the very last line of p. 114 of Maconad's "Symmetric Functions and Hall Polynomials" (2nd ed.). Let's rephrase it in terms of constant terms.

Important Fact: Let

$$\Delta(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j) \quad ,$$

$$p_r(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^r \quad ,$$

Let n be a positive, integer and λ and τ partitions of n. Write $\tau = (\tau_1, \ldots, \tau_k)$ and if λ has $m < n$ parts, define $\lambda_{m+1} = \ldots = \lambda_n = 0$. We have:

$$\chi^\lambda(\tau) \; = \; CT_{x_1,\ldots,x_n} \, \frac{\Delta(x_1,\ldots,x_n) \prod_{i=1}^k p_{\tau_i}(x_1,\ldots,x_n)}{\prod_{i=1}^n x_i^{\lambda_i+n-i}} \quad .$$

Now let's prove Theorem 10.3, part 1. If $|\tau| = n$, then $|\lambda| = n-1$, of course.

Let's consider a typical member of λ^+. It is obtained by changing one of the parts, λ_i to $\lambda_i + 1$. In order for this to be legal, we need $\lambda_{i-1} > \lambda_i$. Doing this gives almost the same expression, but with the constant-termand multiplied by x_i^{-1} (the denominator gets multiplied by x_i). If it is not legal, then $\lambda_{i-1} = \lambda_i$, and the denominator (after multiplying it by x_i) has the degrees of x_{i-1} and x_i being the same. Of course, transposing x_{i-1} and x_i does not change the constant term, but it makes the consant-termand the negative of the original (since $\Delta(x_1,\ldots,x_n)$ is **anti-symmetric**), so the constant term equals to its negative and hence is 0 in this (illegal) case.

Hence, we have to prove that, if $\tau_k \geq 2$, then

$$CT_{x_1,\ldots,x_n} \, \frac{\Delta(x_1,\ldots,x_n) \left(\prod_{i=1}^k p_{\tau_i}(x_1,\ldots,x_n)\right) \cdot \left(\sum_{i=1}^n x_i^{-1}\right)}{\prod_{i=1}^n x_i^{\lambda_i+n-i}} = 0 \quad .$$

But since $\sum_{i=1}^n x_i^{-1} = e_{n-1}(x_1,\ldots,x_n) \cdot \prod_{i=1}^n x_i^{-1}$, the left side equals

$$CT_{x_1,\ldots,x_n} \, \frac{\Delta(x_1,\ldots,x_n) \left(\prod_{i=1}^k p_{\tau_i}(x_1,\ldots,x_n)\right) \cdot e_{n-1}(x_1,\ldots,x_n)}{\prod_{i=1}^n x_i^{\lambda_i+n-i+1}} \quad ,$$

and we have to prove that it equals 0. But since λ is a partition of $n-1$, $\lambda_n = 0$, and we claim the much stronger result

$$CT_{x_n} \, \frac{\Delta(x_1,\ldots,x_n) \left(\prod_{i=1}^k p_{\tau_i}(x_1,\ldots,x_n)\right) \cdot e_{n-1}(x_1,\ldots,x_n)}{\prod_{i=1}^n x_i^{\lambda_i+n-i+1}} = 0 \quad .$$

The **crucial fact** is that the parts of τ are all ≥ 2. Let $Junk_i(x_1,\ldots,x_{n-1})$, $(i = 0,1,2)$ denote various polynomials that only depend on $x_1,\ldots,x_{n-1}$. We have

$$\prod_{i=1}^k p_{\tau_i}(x_1,\ldots,x_n) = Junk_0 + O(x_n^2) \quad ,$$

and we also have:

$$\Delta(x_1,\ldots,x_n) = Junk_1 \cdot \left(\prod_{i=1}^{n-1} (x_i - x_n) \right) = Junk_1 \cdot \left(\left(\prod_{i=1}^{n-1} x_i \right) \cdot x_n^0 \; - \; e_{n-2}(x_1,\ldots,x_{n-1}) x_n + O(x_n^2) \right) \quad .$$

Also

$$e_{n-1}(x_1,\ldots,x_n) = \left(\prod_{i=1}^{n-1} x_i\right)\cdot x_n^0 \;+\; e_{n-2}(x_1,\ldots,x_{n-1})\cdot x_n^1 \quad .$$

Now

$$CT_{x_n}\,[\frac{\Delta(x_1,\ldots,x_n)\left(\prod_{i=1}^{k} p_{\tau_i}(x_1,\ldots,x_n)\right)\cdot e_{n-1}(x_1,\ldots,x_n)}{\prod_{i=1}^{n} x_i^{\lambda_i+n-i+1}}]$$

$$= Junk(x_1,\ldots,x_{n-1})\cdot CT_{x_n}[(Junk_0+O(x_n^2))\cdot$$

$$\left(\left(\left(\prod_{i=1}^{n-1} x_i\right)\cdot x_n^0 \;-\; e_{n-2}(x_1,\ldots,x_{n-1})x_n^1 + O(x_n^2)\right)\cdot\left(\left(\prod_{i=1}^{n-1} x_i\right)\cdot x_n^0 \;+\; e_{n-2}(x_1,\ldots,x_{n-1})x_n^1\right)\cdot\frac{1}{x_n}\right)]$$

and, thanks to $(a-b)(a+b)=a^2-b^2$, this equals:

$$Junk(x_1,\ldots,x_{n-1})\cdot CT_{x_n}[(Junk_0+O(x_n^2))\cdot$$

$$\left(\left(\left(\prod_{i=1}^{n-1} x_i\right)^2\cdot x_n^0 \;-\; (e_{n-2}(x_1,\ldots,x_{n-1}))^2 x_n^2 + O(x_n^2)\right)\cdot\frac{1}{x_n}\right)] \quad .$$

and this is 0. $\square$.