Sharp Upper Bounds for the Orders of The Recurrences Outputted by the Zeilberger and q-Zeilberger Algorithms

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"Er muß sozusagen die Leiter wegwerfen, nachdem er auf ihr hinaufgestiegen ist"
—L. Wittgenstein (Logisch-philosophische Abhandlung, 6.54)

Abstract: We do what the title promises, and as a bonus we get much simplified versions of these algorithms, that do not make any explicit mention of Gosper's algorithm.

Notation. For k integer, $(z)_k := z(z+1)\dots(z+k-1), [a]_k := (1-q^a)(1-q^{a+1})\dots(1-q^{a+k-1})$, if $k \geq 0$ and $(z)_k := 1/(z+k)_{-k}, [a]_k := 1/[a+k]_{-k}$, if $k < 0$. For a *Laurent polynomial* $p(t)$ of t, $deg(p)$ is the *degree*, and $ldeg(p)$ is the *low-degree*, e.g., if $p = 4t^{-3} + 2t^{-1} + 4 + 3t + t^2$, $deg(p) = 2, ldeg(p) = -3$.

Theorem. Let

$$F(n,k) = POL(n,k) \cdot H(n,k) \quad , \qquad (ProperHypergeometric)$$

where $POL(n,k)$ is a polynomial in (n,k) and

$$H(n,k) = \frac{\prod_{j=1}^{A}(a_j'')_{a_j'n+a_jk}\prod_{j=1}^{B}(b_j'')_{b_j'n-b_jk}}{\prod_{j=1}^{C}(c_j'')_{c_j'n+c_jk}\prod_{j=1}^{D}(d_j'')_{d_j'n-d_jk}} z^k \quad , \qquad (PureHypergeometric)$$

where the $a_j, a_j', b_j, b_j', c_j, c_j', d_j, d_j'$ are *non-negative integers*, and z, $a_j'', b_j'', c_j'', d_j''$ are *commuting indeterminates*. We also assume that the factorization in ($ProperHypergeometric$) is *maximal*, i.e. $POL(n,k)$ is of the largest possible degree. Let

$$L = \max\left(\sum_{j=1}^{A} a_j + \sum_{j=1}^{D} d_j \quad , \quad \sum_{j=1}^{B} b_j + \sum_{j=1}^{C} c_j\right) \quad . \qquad (ZBound)$$

There exist polynomials in n, $e_0(n), \dots, e_L(n)$, *not all zero*, and a rational function $R(n,k)$ such that $G(n,k) := R(n,k)F(n,k)$ satisfies

$$\sum_{i=0}^{L} e_i(n)F(n+i,k) = G(n,k+1) - G(n,k) \quad . \qquad (Zpair)$$

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Furthermore, *in general*, L cannot be made any smaller, i.e. the inequality $Order \geq L$ is *sharp*.

Proof: Let

$$\overline{H}(n,k) = \frac{\prod_{j=1}^{A} (a_j'')_{a_j' n + a_j k} \prod_{j=1}^{B} (b_j'')_{b_j' n - b_j k}}{\prod_{j=1}^{C} (c_j'')_{c_j'(n+L)+c_j k} \prod_{j=1}^{D} (d_j'')_{d_j'(n+L)-d_j k}} z^k \quad ,$$

$$f(k) = z \prod_{j=1}^{A} (a_j' n + a_j k + a_j'')_{a_j} \prod_{j=1}^{D} (d_j'(n+L) - d_j k + d_j'' - d_j)_{d_j} \quad ,$$

and

$$g(k) = \prod_{j=1}^{B} (b_j' n - b_j k + b_j'' - b_j)_{b_j} \prod_{j=1}^{C} (c_j'(n+L) + c_j k + c_j'')_{c_j} \quad .$$

Note that $\overline{H}(n,k+1)/\overline{H}(n,k) = f(k)/g(k)$. Write

$$G(n,k) = g(k-1) X(k) \overline{H}(n,k) \quad . \qquad (Ansatz)$$

Substituting into $(Zpair)$ and dividing both sides by $\overline{H}(n,k)$, shows that it is equivalent to

$$f(k)X(k+1) - g(k-1)X(k) - h(k) = 0 \quad , \qquad (Gosper)$$

where

$$h(k) := \sum_{i=0}^{L} e_i(n) POL(n+i,k) \cdot \frac{H(n+i,k)}{\overline{H}(n,k)} \quad .$$

Note that $h(k)$ is a *polynomial* since

$$\frac{H(n+i,k)}{\overline{H}(n,k)} =$$

$$\prod_{j=1}^{A} (a_j' n + a_j k + a_j'')_{i a_j'} \prod_{j=1}^{B} (b_j' n - b_j k + b_j'')_{i b_j'} \prod_{j=1}^{C} (c_j' n + c_j k + c_j'' + i c_j')_{(L-i)c_j'} \prod_{j=1}^{D} (d_j' n - d_j k + d_j'' + i d_j')_{(L-i)d_j'} \quad .$$

We claim that $(Gosper)$ can always be solved (non-trivially) with $X(k)$ being a polynomial of degree $M := deg(h) - max(deg(f), deg(g))$. Writing

$$X(k) = \sum_{i=0}^{M} x_i(n) k^i \quad , \qquad (Ansatz1)$$

substituting into $(Gosper)$, and setting all the coefficients to 0, yields $deg(h) + 1$ *homogeneous* linear equations for the $M + L + 2$ unknowns $e_0(n), \ldots, e_L(n)$, and $x_0(n), \ldots, x_M(n)$. For such a *not-all-zero* solution to exist, we need $\#unknowns - \#equations - 1 \geq 0$, i.e. $(M + L + 2) - (deg(h) + 1) - 1 \geq 0$, i.e. $L \geq max(deg(f), deg(g))$. But

$$deg(f) = \sum_{j=1}^{A} a_j + \sum_{j=1}^{D} d_j \quad , \quad deg(g) = \sum_{j=1}^{B} b_j + \sum_{j=1}^{C} c_j \quad .$$

This concludes the proof *except* that we did not rule out the possibility of $e_0(n), \ldots, e_L(n)$ being all zero (all we are guaranteed, so far, is that it is not possible for *all* of $e_0(n), \ldots, e_L(n)$, *and* $x_0(n), \ldots, x_M(n)$ to be zero). But if all the $e_i(n)$'s are zeros, then $h(k)$ is zero and (*Gosper*) becomes

$$\frac{X(k+1)}{X(k)} = \frac{g(k-1)}{f(k)} \quad .$$

Since $X(k)$ is a polynomial, it means that the roots of $f(k) = 0$ differ from the roots of $g(k-1) = 0$ by *fixed* non-negative integers, which is not possible because of the maximality hypothesis about $POL(n,k)$. Note that the maximality hypothesis always holds, automatically, whenever we have the *generic* situation with z and the $a''_j, b''_j, c''_j, d''_j$ arbitrary (commuting) *symbols.*

To prove that (*Zbound*) is sharp, take $F(n,k) = 1/((1)_k(1)_{n-k})$ and note that L cannot be 0, since otherwise it would have been gosperable with respect to k, but it is not, as can be seen by performing the Gosper algorithm[G] on it. □

q-Theorem. Let

$$F(n,k) = POL(q^n, q^k) \cdot H(n,k) \quad , \qquad (qProperHypergeometric)$$

where $POL(q^n, q^k)$ is a Laurent polynomial in (q^n, q^k), and

$$H(n,k) = \frac{\prod_{j=1}^{A} [a''_j]_{a'_j n + a_j k} \prod_{j=1}^{B} [b''_j]_{b'_j n - b_j k}}{\prod_{j=1}^{C} [c''_j]_{c'_j n + c_j k} \prod_{j=1}^{D} [d''_j]_{d'_j n - d_j k}} q^{Jk(k-1)/2} z^k \quad , \qquad (qPureHypergeometric)$$

where the $a_j, a'_j, b_j, b'_j, c_j, c'_j, d_j, d'_j$ are *non-negative integers*, and z, $a''_j, b''_j, c''_j, d''_j$ are *indeterminates*, and J is an integer. We also assume that the factorization in (*qProperHypergeometric*) is *maximal*, i.e. $POL(q^n, q^k)$ is as 'large' as possible. Let

$$L = \max\left(J + \sum_{j=1}^{A} {a_j}^2 \ , \ \sum_{j=1}^{C} {c_j}^2 \right) + \max\left(-J + \sum_{j=1}^{D} {d_j}^2 \ , \ \sum_{j=1}^{B} {b_j}^2 \right) \quad . \qquad (qZBound)$$

There exist polynomials in q^n, $e_0(q^n), \ldots, e_L(q^n)$, *not all zero*, and a rational function $R(q^n, q^k)$ such that $G(n,k) := R(q^n, q^k) F(n,k)$ satisfies

$$\sum_{i=0}^{L} e_i(q^n) F(n+i,k) = G(n,k+1) - G(n,k) \quad . \qquad (qZpair)$$

Furthermore, *in general*, L, cannot be made any smaller, i.e. the inequality $Order \geq L$ is *sharp.*

Proof: Let

$$\overline{H}(n,k) = \frac{\prod_{j=1}^{A} [a''_j]_{a'_j n + a_j k} \prod_{j=1}^{B} [b''_j]_{b'_j n - b_j k}}{\prod_{j=1}^{C} [c''_j]_{c'_j (n+L) + c_j k} \prod_{j=1}^{D} [d''_j]_{d'_j (n+L) - d_j k}} q^{Jk(k-1)/2} z^k \quad ,$$

$$f(k) = zq^{Jk} \prod_{j=1}^{A} [a'_j n + a_j k + a''_j]_{a_j} \prod_{j=1}^{D} [d'_j(n+L) - d_j k + d''_j - d_j]_{d_j} \quad ,$$

and

$$g(k) = \prod_{j=1}^{B} [b'_j n - b_j k + b''_j - b_j]_{b_j} \prod_{j=1}^{C} [c'_j(n+L) + c_j k + c''_j]_{c_j} \quad .$$

Note that $\overline{H}(n,k+1)/\overline{H}(n,k) = f(k)/g(k)$. Write

$$G(n,k) = g(k-1)X(k)\overline{H}(n,k) \quad . \qquad (qAnsatz)$$

Substituting into $(Zpair)$ and dividing both sides by $\overline{H}(n,k)$, shows that it is equivalent to

$$f(k)X(k+1) - g(k-1)X(k) - h(q^k) = 0 \quad , \qquad (qGosper)$$

where

$$h(q^k) := \sum_{i=0}^{L} e_i(n) POL(q^n q^i, q^k) \cdot \frac{H(n+i,k)}{\overline{H}(n,k)} \quad .$$

Note that $h(q^k)$ is a *Laurent polynomial* (in q^k) since

$$\frac{H(n+i,k)}{\overline{H}(n,k)} =$$

$$\prod_{j=1}^{A} [a'_j n + a_j k + a''_j]_{ia'_j} \prod_{j=1}^{B} [b'_j n - b_j k + b''_j]_{ib'_j} \prod_{j=1}^{C} [c'_j n + c_j k + c''_j + ic'_j]_{(L-i)c'_j} \prod_{j=1}^{D} [d'_j n - d_j k + d''_j + id'_j]_{(L-i)d'_j} \quad .$$

Let

$$M_1 := -ldeg(h) - max(-ldeg(f), -ldeg(g)) \quad , \quad M_2 := deg(h) - max(deg(f), deg(g)) \quad .$$

We claim that $(qGosper)$ can always be solved (non-trivially) with $X(k)$ a Laurent polynomial of q^k of low-degree $-M_1$ and degree M_2. Writing

$$X(k) = \sum_{i=-M_1}^{M_2} x_i(n)(q^k)^i \quad , \qquad (qAnsatz1)$$

substituting into $(qGosper)$, and setting all the coefficients to 0, yields $-ldeg(h)+deg(h)+1$ *homogeneous* linear equations for the M_1+M_2+L+2 unknowns $e_0(n), \ldots, e_L(n)$, and $x_{-M_1}(n), \ldots, x_{M_2}(n)$. For such a *not-all-zero* solution to exist, we need $\#\, unknowns \,-\, \#\, equations\, -1 \geq 0$, i.e. $(M_1+M_2+L+2)-(-ldeg(h)+deg(h)+1)-1 \geq 0$, i.e. $L \geq max(deg(f), deg(g)) + max(-ldeg(f), -ldeg(g))$. But

$$deg(f) = J + \sum_{j=1}^{A} {a_j}^2 \quad , \quad -ldeg(f) = -J + \sum_{j=1}^{D} {d_j}^2 \quad , \quad deg(g) = \sum_{j=1}^{C} {c_j}^2 \quad , \quad -ldeg(g) = \sum_{j=1}^{B} {b_j}^2 \quad .$$

This concludes the proof *except* that we did not rule out the possibility of $e_0(n), \ldots, e_L(n)$ being all zero (all we are guaranteed, so far, is that it is not possible for *all* of $e_0(n), \ldots, e_L(n)$, *and* $x_{-M_1}(n), \ldots, x_{M_2}(n)$ to be zero). But if all the $e_i(n)$'s are zeros, then $h(q^k)$ is zero and ($qGosper$) becomes

$$\frac{X(k+1)}{X(k)} = \frac{g(k-1)}{f(k)} \quad .$$

Since $X(k)$ is a Laurent polynomial in q^k, it means that the roots of $f(k) = 0$ differ from the roots of $g(k-1) = 0$ by *fixed* non-negative integers, which is not possible because of the maximality hypothesis about $POL(q^n, q^k)$. Note that the maximality hypothesis always holds, automatically, whenever we have the *generic* situation with z and the $a''_j, b''_j, c''_j, d''_j$ arbitrary *symbols*.

To prove that ($qZbound$) is sharp, take $F(n,k) = q^{k(k-1)/2}/([1]_k[1]_{n-k})$, and note that L cannot be 0, since otherwise it would have been q-gosperable with respect to k, but it is not, as can be seen by performing the q-Gosper algorithm ([Kor][PR], or use `qEKHAD`) on it. □

Comments

1. The bounds in ($ZBound$) and ($qZBound$) considerably improve those of [WZ] (Theorems 3.1 and 5.1, see also [PWZ] and [Koe]), that relied on Sister Celine's Technique, and, as we proved, are *sharp* for the *generic* case. However, sometimes a system of linear equations with more equations than unknowns *does* have a non-trivial solution, and also, sometimes one can find higher-degree polynomial solutions to ($Gosper$) and ($qGosper$), so in *specific* cases, it is possible to have recurrences of lower order. This is the case for all the non-trivial classical hypergeometric (binomial-coefficient) sums that admit a closed-form evaluation.

2. The proofs imply new, simplified, versions of the Zeilberger[Z1][Z2][PS] and q-Zeilberger[Kor][PR] algorithms. These new versions do not rely on Gosper's algorithm *explicitly*, but, of course, were inspired by it. In fact, they were designed by applying the Zeilberger and q-Zeilberger algorithms *once and for all*, to the *generic* cases. It so happens, that in this case, a simplified version of Gosper (and q-Gosper) suffices, and it is so simple that it can be incorporated implicitly. So old-Zeilberger (and hence Gosper and q-Gosper) is the Wittgensteinian ladder that we *must* throw away after we climbed it.

The *running-time* complexity of these new versions are comparable to the old versions, but their *program-length complexity* (in the sense of Chaitin-Kolmogorov) are considerably smaller

The simplified Zeilberger and q-Zeilberger algorithms, apply also to *specific*, non-generic summands. Start by taking $L = 0$ and try the *ansatzes* ($Ansatz$) and ($qAnsatz$), but with M (for the q-case: M_1, M_2)) possibly larger than the ones in the theorem (which are determined by plugging them into ($Gosper$) or ($qGosper$), and equating the leading coefficient(s), and finding out whether they can vanish for integral M (or M_1, M_2). Then one solves the resulting set of linear equations. If there is no non-trivial solution, then, one increases L by 1, until success is reached. The theorems guarantee that eventually we will succeed, at worst, with the L's given by ($ZBound$) and ($qZBound$).

3. These simplified versions are implemented in the Maple packages `ZEILBERGER` and `qZEILBERGER` available from **http://www.math.rutgers.edu/~zeilberg/Zeilberger/sharp.html**, where one can also find several examples.

4. The present article was intentionally written in a terse, unmotivated, style, in order to emphasize its simplicity and self-containedness. Readers who wish to see more motivation are welcome to look at an earlier version, that only covers the ordinary case, that is also available from the above webpage.

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