Investigating the Sahi Property

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In [S], Lemma 16, it is proved, using ad-hoc, human methods, that if $\alpha, \beta, \gamma, \delta$, are non-negative and $\alpha+\beta+\gamma+\delta=1$ and $\alpha\,\delta \geq \beta\,\gamma$, then the following formal power series

$$1-(1-a)^{\delta}\,(1-a-b-c)^{\gamma}\,(1-a-b-d)^{\beta}\,(1-a-b-c-d-e)^{\alpha} \quad ,$$

is a formal power series in the variables a, b, c, d, e with **non-negative** coefficients.

For now, for the sake of simplicity, let's take $\alpha=\beta=\gamma=\delta=\frac{1}{4}$, so we have, thanks to Sahi the fact that

$$1-((1-a)(1-a-b-c)(1-a-b-d)(1-a-b-c-d-e))^{\frac{1}{4}} \quad ,$$

is a formal power series in the variables a, b, c, d, e with **non-negative** coefficients.

Let $x_1, \ldots, x_n$ be formal variables and let $S = \{\sum_{j=1}^{n} c_{ij} x_i, i = 1..m)\}$ be a set of linear combinations of the variables. Let α be a number, the pair (S, α) has the (n,m)-**Sahi property** if the formal power series

$$1-\prod_{1}^{m}(1-\sum_{j=1}^{n} c_{ij} x_i)^{\alpha} \quad ,$$

is a formal power series with **non-negative** coefficients.

So Sahi's Lemma 16 says that the pair $(\{x_1, x_1+x_2+x_3, x_1+x_2+x_4, x_1+x_2+x_3+x_4+x_5\}, \frac{1}{4})$ has the $(5,4)$-Sahi property.

Big Challenge For given m and n find, experimentally, and if possible rigorously, all pairs (S, α) with the Sahi property.

We have to start with the first not completely trivial case, $n=1$ and $m=2$.

using the Maple program `SahiProperty.txt` it is seen that $\{x, x/4\}, \frac{7}{10})$ does **not** have the Sahi property, but $\{x, x/4\}, \frac{3}{5})$ does.

First mini-challenge

Find all pairs $(c\alpha)$ such that the one-variable formal power series

$$1-((1-x)\,(1-c\,x))^{\alpha} \quad ,$$

is a formal power series with non-negative coefficients.

References [S] Siddhartha Sahi, *Higher correlation inequalities*, Combinatorica **28** (2008), 209-227.

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