A Multi-set Identity for Partitions

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Guo-Niu Han kindly pointed out to us (something that we should have noticed ourselves if we would have been in the habit of reading **carefully** all the papers that we cite), that our main result is contained in [B.H.].

Introduction

Given an integer-partition $\lambda \vdash n$ and a box (a cell) $v = [i,j] \in \lambda$ it determines the arm length a_v $(= \lambda_i - j)$, the leg length l_v $(=\lambda'_j - i)$, and the left length f_v $(= j-1)$. Thus, for example, the hook length h_v is given by $h_v = a_v + l_v + 1$. Denote $p_v = a_v + f_v + 1$. C. Bessenrodt [B], and R. Bacher and L. Manivel [B.M] (see also [B.H]) proved the following identity:

$$(1) \qquad \sum_{\lambda \vdash n} \sum_{v \in \lambda} x^{h_v} = \sum_{\lambda \vdash n} \sum_{v \in \lambda} x^{p_v},$$

which is equivalent to the multi-set identity:

$$(2) \qquad \bigcup_{\lambda \vdash n} \{h_v \mid v \in \lambda\} = \bigcup_{\lambda \vdash n} \{p_v \mid v \in \lambda\}.$$

In this note we prove the following refinement of (2).

Fill v with a pair of numbers in two different ways:

First Filling: Fill v with (a_v, l_v).

Second Filling: Fill v with (a_v, f_v).

This yields the following two multi-sets of pairs:

$$A_1(n) = \bigcup_{\lambda \vdash n} \{(a_v, l_v) | v \in \lambda\} \quad ,$$

$$A_2(n) = \bigcup_{\lambda \vdash n} \{(a_v, f_v) | v \in \lambda\} \quad .$$

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Theorem 1: For all non-negative integers n we have the multi-set identity,

$$A_1(n) = A_2(n).$$

The proof here is by applying the technique of generating functions. Theorem 1 indicates that for each n there is a map φ on the cells of the partitions of n, $\varphi : v \to \varphi(v)$, such that $(a_v, f_v) = (a_{\varphi(v)}, l_{\varphi(v)})$. The construction of an explicit such φ – for all n – would yield a bijective proof of Theorem 1.

The proof.

As usual, $(z)_a := (1-z)(1-qz)\cdots(1-q^{a-1}z)$.

The proof would follow from the following two lemmas.

Lemma 1: Let $M_1(c,d)(n)$ be the number of times the pair (c,d) shows up in $A_1(n)$, then

$$\sum_{n=0}^{\infty} M_1(c,d)(n) q^n = \frac{q^{c+d+1}}{1-q^{c+d+1}} \cdot \frac{1}{(q)_\infty} \quad . \tag{1}$$

Lemma 2: Let $M_2(c,d)(n)$ be the number of times the pair (c,d) shows up in $A_2(n)$, then

$$\sum_{n=0}^{\infty} M_2(c,d)(n) q^n = \frac{q^{c+d+1}}{1-q^{c+d+1}} \cdot \frac{1}{(q)_\infty} \quad . \tag{2}$$

Proof of Lemma 2: $M_2(c,d)(n)$ counts the number of Ferrers diagrams of n where one of the cells that has (right) arm c and left-arm d is **marked**. Obviously it belongs to a row of length $c+d+1$, and each such row has exactly one such cell. Hence this is the same as counting the number of Ferrers diagrams of n where one of the rows of length $c+d+1$ is marked. We can construct such a Ferrers diagram (with any number of cells) by first drawing that row of length $c+d+1$ (weight q^{c+d+1}) then putting **below** it an arbitrary Ferrers diagram with largest part $\leq c+d+1$, whose generating function is $1/((1-q)(1-q^2)\cdots(1-q^{c+d+1}))$, and then placing **above** the above-mentioned fixed row any Ferrers diagram whose *smallest* part is $\geq c+d+1$, whose generating function is $1/((1-q^{c+d+1})(1-q^{c+d+2})\cdots)$. Combining, we get that the generating function of such marked creatures, which is the left side of (2), is the right side of (2) $\square$.

Before proving Lemma 1 we have to recall certain basic facts from q-land.

Fact 1 (The q-Binomial Theorem [essentially Theorem 2.1 of [A][3]]).

$$\frac{1}{(z)_{a+1}} = \sum_{j=0}^{\infty} \frac{(q)_{a+j}}{(q)_a (q)_j} z^j \quad .$$

[3] But the "conditions" $|q| < 1, |t| < 1$, stated by Andrews, are, in our world-view, a *category mistake*.

(This is easily proved by induction on a).

When $a = \infty$ this simplifies to

Fact 2

$$\frac{1}{(z)_\infty} = \sum_{j=0}^{\infty} \frac{z^j}{(q)_j} \quad .$$

Fact 3: The generating function for Ferrers diagrams bounded in an m by n rectangle is $\frac{(q)_{m+n}}{(q)_m (q)_n}$.

This is Proposition 1.3.19 in [St] and Theorem 3.1 of [A]. Here is a proof by induction of this elementary fact. Let the generating function be $F(m, n; q)$. Consider the last cell of the top row. If it is occupied, the generating function of these diagrams is $q^n F(m-1, n)$ (remove the fully-occupied top row), if it is not, it is $F(m, n-1)$ (delete the empty rightmost column), getting the recurrence $F(m, n; q) = q^n F(m-1, n; q) + F(m, n-1; q)$. Then verify that the same recurrence is satisfied by $\frac{(q)_{m+n}}{(q)_m (q)_n}$, and check the trivial initial conditions $m = 0$ and $n = 0$.

By sending n to infinity we obtain

Fact 4: The generating function for Ferrers diagrams with parts bounded by m is $\frac{1}{(q)_m}$. By conjugation, this is also the generating function for Ferrers diagrams with at most m parts.

Proof of Lemma 1: The left-side of (1) is the generating function for Ferrers diagrams where one **hook** with arm-length c and leg-length d is **marked**. Let's figure out the generating function (weight-enumerator) for all such (c, d)-hook-marked Ferrers diagrams.

Suppose the corner of that hook is at cell $(i+1, j+1)$ (i.e. the $(i+1)$-row and the $(j+1)$-column). Here $0 \leq i < \infty$ and $0 \leq j < \infty$. Let's look at its *anatomy*. It consists of **seven parts**. (See diagram in
http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/TemunaFerrers.html).

1. Strictly **left of** *and* **above** cell $(i+1, j+1)$. This is a fully occupied i by j rectangle with weight q^{ij}.

2. Above the arm (of length $c+1$). This is a fully occupied i by $c+1$ rectangle with weight $q^{(c+1)i}$.

3. To the left of the leg (of length $d+1$). This is a fully occupied $d+1$ by j rectangle with weight $q^{(d+1)j}$.

4. The Ferrers diagram with $\leq i$ rows lying **above and to the right** of the arm. By Fact 4, the generating function of this is $1/(q)_i$.

5. The Ferrers diagram with $\leq j$ columns lying **below and to the left** of the leg. By Fact 4, the generating function of this is $1/(q)_j$.

6. The hook itself. This gives generating function q^{c+d+1}.

7. The Ferrers diagram formed **inside** the hook, i.e. lying below the arm and to the right of the leg. By Fact 3 its generating function is $\frac{(q)_{c+d}}{(q)_c(q)_d}$.

Combining, we see that the generating function for these (c, d)-hook-marked Ferrers diagrams is

$$\frac{(q)_{c+d}}{(q)_c(q)_d} \cdot q^{c+d+1} \cdot q^{ij+i(c+1)+j(d+1)} \cdot \frac{1}{(q)_i} \cdot \frac{1}{(q)_j} \quad .$$

Summing over **all** $0 \le i, j < \infty$, we get that the generating function on the left of (1) equals

$$\frac{(q)_{c+d}}{(q)_c(q)_d} q^{c+d+1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q^{ij+i(c+1)+j(d+1)} \frac{1}{(q)_i} \frac{1}{(q)_j}$$

$$= \frac{(q)_{c+d}}{(q)_c(q)_d} q^{c+d+1} \sum_{i=0}^{\infty} \frac{1}{(q)_i} q^{i(c+1)} \sum_{j=0}^{\infty} q^{j(i+d+1)} \frac{1}{(q)_j}$$

$$= \frac{(q)_{c+d}}{(q)_c(q)_d} q^{c+d+1} \sum_{i=0}^{\infty} \frac{1}{(q)_i} q^{i(c+1)} \frac{1}{(q^{d+i+1})_\infty} \quad ,$$

by Fact 2 with $z = q^{d+i+1}$. This, in turn, equals

$$\frac{(q)_{c+d}}{(q)_c} q^{c+d+1} \sum_{i=0}^{\infty} q^{i(c+1)} \frac{1}{(q)_\infty} \frac{(q^{i+1})_d}{(q)_d}$$

$$= \frac{q^{c+d+1}}{(q)_\infty} \frac{(q)_{c+d}}{(q)_c} \sum_{i=0}^{\infty} q^{i(c+1)} \frac{(q)_{i+d}}{(q)_d(q)_i}$$

$$= \frac{q^{c+d+1}}{(q)_\infty} \frac{(q)_{c+d}}{(q)_c} \frac{1}{(q^{c+1})_{d+1}} \quad ,$$

by Fact 1 with $z = q^{c+1}$. Finally, this equals

$$= \frac{q^{c+d+1}}{(q)_\infty} \cdot \frac{(1-q)(1-q^2)\cdots(1-q^{c+d})}{(1-q)(1-q^2)\cdots(1-q^{c+d+1})} = \frac{q^{c+d+1}}{(q)_\infty} \cdot \frac{1}{(1-q^{c+d+1})} \quad \Box.$$

Reference

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