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Note

A Purely Verification Proof of the First Rogers–Ramanujan Identity

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We will give a new proof of the following identity of G. Andrews [A1], see also [B]), which immediately implies the first Rogers–Ramanujan identity upon taking $n \to \infty$, and using the Jacobi triple product identity. (Throughout this paper $s = q^n$, $t = q^k$, $(q)_a = (1-q)(1-q^2)\cdots(1-q^a)$, for $a \geqslant 0$, and $1/(q)_a = 0$ for $a < 0$.)

THEOREM [A1]. *For every non-negative integer n we have*

$$\sum_k \frac{q^{k^2}}{(q)_k (q)_{n-k}} = \sum_k \frac{(-1)^k q^{(5k^2-k)/2}}{(q)_{n-k}(q)_{n+k}}. \qquad \text{(FRR)}$$

Let $L(n)$ be the left side and $R(n)$ be the right side of (FRR). It is readily checked that $L(n) = R(n)$ for $n = 0, ..., 4$. We will show that both $L(n)$ and $R(n)$ are solutions of a certain fifth order linear recurrence equation, and the result will then follow by induction. Let N be the shift operator in the n direction: $Nf(n) = f(n+1)$. Let

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$$P_1(s, q, N) = q - (1 - sq^2 + s^2q^3 + q)N + (1 - sq^2)N^2,$$

$$\begin{aligned} P_2(s, q, N) = {} & q^9 - q^5(1 + q^2 + q + s^2q^8 + sq^5 + q^7s^2)N \\ & + q^2(1 + q + q^2 + sq^5 + s^3q^{13} + q^6s + s^4q^{16} + s^2q^8)N^2 \\ & + (sq^5 + 1)(q^9s^2 - 1)N^3. \end{aligned}$$

(i) $P_1(s, q, N)\, L(n) = 0$, and thus $P_2(s, q, N)\, P_1(s, q, N)\, L(n) = 0$.

Proof of (i). Let $F_1(n, k)$ be the summand on the left side of (FRR). We cleverly construct $G_1(n, k) = s^2tq^2F_1(n, k)/(sq - t)$, with the motive that $P_1(s, q, N)\, F_1(n, k) = G_1(n, k) - G_1(n, k-1)$, (check!) and (i) follows by summing w.r.t. k. ∎

(ii) $P_2(s, q, N)\, P_1(s, q, N)\, R(n) = 0$.

Proof of (ii). Let $F_2(n, k)$ be the summand on the right side of (FRR). We cleverly construct

$$G_2(n, k) = \frac{(1 - stq^5 + sq^6t^4 - q^2t^5)\, s^5q^{14}t^4 \cdot F_2(n, k)}{\left(\begin{array}{c}(1 - stq^5)(1 - stq^4)(1 - stq^3)(1 - stq^2) \\ (1 - stq)(t - sq^4)(sq^3 - t)(sq^2 - t)(sq - t)\end{array}\right)},$$

with the motive that

$$P_2(q, s, N)\, P_1(q, s, N)\, F_2(n, k) = G_2(n, k) - G_2(n, k-1), \text{ (check!)},$$

and(ii) follows upon summing with respect to k. ∎

Remarks. (i) The above proof was independently generated by the two authors by executing a MAPLE program, that is based on the algorithm in [Z1] for proving terminating q-hypergeometric (alias q-binomial) identities. The program is available from Doron Zeilberger upon request. Using this program we were also able to prove the second Rogers–Ramanujan identity.

(ii) (FRR) was discovered, and proved, by George Andrews [A1] by observing that it is a limiting case of Watson's q-analog of Whipple's transformation. He asked for an elementary proof that was given by David Bressoud [B]. The program of [Z1] should also, in principle, be able to prove the full q-Whippie transformation, but our memories did not suffice, although we did prove the ordinary Whipple transformation using the algorithm of [Z2].

(iii) The title of this note is an allusion to Hardy's criticism of most of the proofs of the Rogers–Ramanujan identities as being "essentially verifications". A conceptual proof was recently given by Leon Ehrenpreis [E], and his method was further used by Friedman [F] to get nice analytical proofs of results of Andrews [A2].

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