Automatic Solving of Cubic Diophantine Equations Inspired by Ramanujan

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Abstract. In Ramanujan's Lost Notebook, he stated infinitely many 'almost counterexamples' to Fermat's Last Theorem for $n = 3$, by solving $X^3 + Y^3 - Z^3 = 1$. As often the case with Ramanujan, he gave no indication how he discovered it. Using ingenious *exegesis* of Ramanujan's mind by Michael Hirschhorn, combined with a much earlier 'reading of Ramanujan's mind' by Eri Jabotinsky, we automate this process, and develop a symbolic-computational algorithm, based on the C-finite ansatz, to solve much more general equations, namely cubic equations of the form $a\,X^3 + a\,Y^3 + b\,Z^3 = c$.

Preface: How this paper was born

2020: On Sunday, May 24, 2020 we were fortunate to attend a surprise retirement party for our dear friend Dennis Stanton, and one of the eight speeches was by the eminent combintorialist and number theorist, George Andrews, who presented Dennis a retirement present, the book [Al] *"The Boy Who Dreamed of Infinity"* authored by his daughter, Amy Alznauer, meant for "the children in Dennis' life". As soon as the party was over, we ordered a copy and after a few days got it. We *loved* this book. It turned out that this book can be enjoyed by **all** ages from 0 to ∞. One of the delightful illustrations, contained the following **amazing** identity, taken from *Ramanujan's Lost Notebook* [R] (p. 341).

Theorem (Ramanujan): Define sequences of integers a_n, b_n, c_n, $n \geq 0$, in terms of the following generating functions

$$\sum_{n\geq 0} a_n x^n = \frac{1+53x+9x^2}{1-82x-82x^2+x^3} \quad ,$$

$$\sum_{n\geq 0} b_n x^n = \frac{2-26x-12x^2}{1-82x-82x^2+x^3} \quad ,$$

$$\sum_{n\geq 0} c_n x^n = \frac{2+8x-10x^2}{1-82x-82x^2+x^3} \quad ,$$

then

$$a_n^3 + b_n^3 = c_n^3 + (-1)^n \quad . \tag{1}$$

This theorem is very attractive since it furnishes infinitely many *almost-counterexamples* to Fermat's last theorem for $n = 3$. It turns out that we have seen this identity before, so let's backtrack 25 years.

1995: As with many of Ramanujan's theorems, he did not give a proof, and left it to posterity (most notably Bruce Berndt and George Andrews and their students) to furnish proofs. In 1995, Michael Hirschhorn gave a proof [Ha1] (see also [Ha2] and [HaHi]) and more interestingly, indicated how Ramanujan may have discovered it. Hirschhorn's *exegesis* started with a **polynomial identity** that appeared in Ramanujan's notebooks.

$$(A^2+7AB-9B^2)^3+(2A^2-4AB+12B^2)^3+(-2A^2-10B^2)^3+(-A^2+9AB+B^2)^3 \;=\; 0 \quad . \; (2)$$

Hirschhorn speculated that this polynomial identity was the starting point of Ramanujan's discovery. Then he solved the quadratic diophantine equation

$$A^2-9AB-B^2=\pm 1 \quad ,$$

in terms of the recursive sequence h_n defined by $h_0=0, h_1=1$ and $h_{n+2}=9h_{n+1}+h_n$ and showed that indeed it is satisfied by $A=h_{n+1}$ and $B=h_n$. Then Hirschhorn (and presumably Ramanujan) found generating functions for the remaining quantities $A^2+7AB-9B^2$, $2A^2-4AB+12B^2$ and $2A^2+10B^2$.

As noticed by Hirschhorn, once Ramanujan's amazing identity is conjectured, its formal proof is **purely routine**. In fact (see [Ha2]) checking it for the seven special cases $n=0,1,2,3,4,5,6$ consists a **fully rigorous proof**. As we will explain later, the **broader context** is the C-finite ansatz (Ch. 4 of [KP]; [Z2]).

But how in the world did Ramanujan come up with the equally amazing *polynomial* identity (2)? In this case it is even more obvious that **once conjectured**, the proof is a routine high-school-algebra computation, but how did Ramanujan find these four quadratic polynomials
$P_1(A,B), P_2(A,B), P_3(A,B), P_4(A,B)$ with integer coefficients such that

$$P_1(A,B)^3+P_2(A,B)^3+P_3(A,B)^3+P_4(A,B)^3 \;=\; 0 \quad ?$$

Now we have to backtrack further in time, about 55 years ago.

1965: One of us (DZ) remembered that when he was fifteen-years-old he was fascinated by an article [J], published in his then (and still now) favorite journal "gilyonot letmatemtika" [in Hebrew] by Eri Jabotinsky(1910-1969) in which he attempted to "read Ramanujan's mind" and explain how he may have come up with such infinite families of 'sum of cubes identities' generalizing $3^3+4^3+5^3=6^3$. The details were forgotten, but thanks to the efforts of Gadi Aleksandrowicz, from the Technion, Israel, it has been archived and made available here

`https://gadial.github.io/netgar/` .

We easily were able to locate Jabotinsky's article, that so fascinated us 55 years ago, and use it for this project. Jabotinsky's 'trick' will be explained later.

Why this article

Now that we know Ramanujan's 'tricks' (at least as conjectured by Hirschhorn and Jabotinksy, and it does not matter whether that's how Ramanujan actually did it), can they be used for other problems? Can they be turned into an algorithm that can be taught to a computer? Using computer algebra, we were able to combine these two 'tricks' and developed an algorithm to solve diophantine equations of the more general form

$$a\,X^3 + a\,Y^3 + b\,Z^3 \; = \; c \quad ,$$

for *all* integers a, b and a select set of c's.

Sample Theorem: The front of this article
`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/ramacubic.html`
contains lots of theorems. For example the output file
`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oRamanujanCubic1.txt`
contains 641 theorems similar to Ramanujan's theorem above. Let's just reproduce one here, as an appetizer.

Theorem (S.B. Ekhad): Define sequences of integers a_n, b_n, c_n, $n \geq 0$, in terms of the following generating functions

$$\sum_{n\geq 0} a_n t^n \; = \; \frac{293155\,t^2 + 888826\,t - 29}{(1-t)\,(t^2 - 103682\,t + 1)} \quad ,$$

$$\sum_{n\geq 0} b_n t^n \; = \; -\frac{237169\,t^2 + 550798\,t + 1}{(1-t)\,(t^2 - 103682\,t + 1)} \quad ,$$

$$\sum_{n\geq 0} c_n t^n \; = \; \frac{90601\,t^2 - 878594\,t + 25}{(1-t)\,(t^2 - 103682\,t + 1)} \quad ,$$

then

$$a_n^3 + 2b_n^3 + 2c_n^3 \; = \; 6859 \quad . \tag{1}$$

The Maple packages

This article is accompanied by three Maple packages.

• `RamanujanCubic.txt` : This is the main Maple package. It can automatically solve diophantine equation of the from $a\,X^3 \;+\; a\,Y^3 \;+\; b\,Z^3 \;=c$ for any given choice of integers a, b and some c (that depend on a and b) by furnishing a *parametric solution* thereby furnishing *infinitely* many solutions.

• `Pell.txt`: It solves Pell, and Pell-like diophantine equations for quadratic binary forms, using symbolic computation.

• `RatDio.txt`: It uses the C-finite ansatz to manufacture complicated diophantine equations that are known to have solutions beforehand, and also solves them in the quadratic case, using the C-finite ansatz.

They are all available from the front mentioned above, along with numerous sample input and output files.

Diophantine Equations In General

Hilbert's dream, expressed in his 10th problem, to find an algorithm that would decide whether or not *any* diophantine equation is solvable was famously shattered by Yuri Matiyashevich, standing on the shoulders of Julia Robinson, Martin Davis, and Hilary Putnam. But for many *families*, e.g. Pell's equation $x^2 - N y^2 = 1$ (for any non-square) N , and Fermat's $x^n + y^n - z^n = 0$ ($n > 2$) it was possible to *decide*, positively and negatively, respectively. But let us remark that by going *backwards*, and with the aid of computer algebra, we can easily **concoct** many diophantine equations that are *guaranteed* to have infinitely many solutions.

Recall that, famously, the equation $x^2 + y^2 = z^2$ has the *parametric* solution

$$x = m^2 - n^2 \quad , \quad y = 2mn \quad , \quad z = m^2 + n^2 \quad .$$

Let's start with any polynomials (with integer coefficients) $P(m, n)$, $Q(m, n)$ and $R(m, n)$, and define

$$x = P(m, n) \quad , \quad y = Q(m, n) \quad , \quad z = R(m, n) \quad .$$

Eliminating m, n from these equations (using, most efficiently, the Buchberger algorithm) we immediately find a polynomial (with integer coefficients) $S(x, y, z)$ such that $S(P(m, n), Q(m, n), R(m, n))$ is identically 0. So we can cook-up many theorems.

Suppose that, in a *counterfactual world*, you never heard of Pythagorean triples, and defined, out of the blue, $x = m^2 - n^2 \quad , \quad y = 2mn \quad , \quad z = m^2 + n^2$, and wondered how x, y, z are related *implicitly*. If you enter, in Maple:

```
Groebner[Basis](x-(m**2-n**2),y-2*m*n,z-m**2-n**2,plex(m,n,x,y,z))[1];    ,
```

you would get the beautiful relationship

$$x^2 + y^2 - z^2 = 0 \quad .$$

Typing instead

```
Groebner[Basis](x-(2*m**2-3*n**2),y-2*m*n,z-m**2-n**2,plex(m,n,x,y,z))[1];
```

gives the less good-looking relationship

$$4\,x^2 + 4\,xz + 25\,y^2 - 24\,z^2 \;=\; 0 \quad ,$$

but you can publish a paper with the next deep theorem.

Theorem: The diophantine equation

$$4\,x^2 + 4\,xz + 25\,y^2 - 24\,z^2 \;=\; 0 \quad ,$$

has (doubly) infinite many solutions given by the parametric equation

$$x = 2m^2 - 3n^2 \quad , \quad y = 2mn \quad , \quad z = m^2 + n^2 \quad .$$

Proof: (check!).

This way one can make up many such deep theorems. Here is another example

`Groebner[Basis](x-(m**3-n**3),y-m**2*n-m*n**2,z-(m**3+n**3),plex(m,n,x,y,z))[1];` gives

$$3\,x^2y + x^2z + 4\,y^3 - 3\,yz^2 - z^3 \quad ,$$

and we immediately have the following deep theorem.

Theorem: The diophantine equation

$$3\,x^2y + x^2z + 4\,y^3 - 3\,yz^2 - z^3 \;= 0 \quad ,$$

has doubly-infinite many solutions given by the parametric equation

$$x = m^3 - n^3 \quad , \quad y = m^2n + mn^2 \quad , \quad z = m^3 + n^3 \quad .$$

Proof: (check!).

You are welcome to create your own deep theorems, for which the equation has (doubly!) infinitely many solutions.

To create random examples for which we know a-priori that there are **no** solutions is a bit harder, but you can "cheat" and take any diophantine equation, e.g. $x^3 + y^3 + z^3 = 0$ for which it has been already proven that there exist no (non-trivial) solutions. Then replace x, y and z by arbitrary expressions. For example, typing, in a Maple session,

```
expand(subs(x = 6*x+7*y-9*z, y = 6*x-5*y+4*z, z = -8*x-3*y+3*z,x**3+y**3+z**3));
```

gives

$$-80\,x^3 - 360\,x^2y + 36\,x^2z + 1116\,xy^2 - 2556\,xyz + 1530\,xz^2 + 191\,y^3 - 942\,y^2z + 1380\,yz^2 - 638\,z^3$$

and we immediately have

Theorem: The diophantine equation

$$-80\,x^3 - 360\,x^2y + 36\,x^2z + 1116\,xy^2 - 2556\,xyz + 1530\,xz^2 + 191\,y^3 - 942\,y^2z + 1380\,yz^2 - 638\,z^3 \;=\; 0$$

has no (non-trivial) solutions.

Creating Diophantine Equations using the C-finite ansatz

Another way to **start with the answer** is to use the C-finite ansatz ([KP],ch. 4; [Z2]).

Let's start with quadratic equations. We have the following theorem, that once stated is easy to prove in the context of the C-finite ansatz.

General Theorem: Let $c0, c1,\ d0, d1$ and k be integers and define two sequences of integers $a(n), b(n)$

$$\sum_{n=0}^{\infty} a(n)x^n \ = \ \frac{c0+c1x}{1-kx+x^2} \quad ,$$

$$\sum_{n=0}^{\infty} b(n)x^n \ = \ \frac{d0+d1x}{1-kx+x^2} \quad ,$$

then there exists a homogeneous binary quadratic form $P(X,Y)$, with integer coefficients, and an integer C, such that for all $n \geq 0$

$$P(a(n),b(n)) \ = \ C \quad .$$

In fact, more explicitly

$$\left(c0\ d1 - c1\ d0\right)^2 \left(d0\ d1\ k + d0^2 + d1^2\right) a(n)^2$$
$$-\left(c0\ d1 - c1\ d0\right)^2 \left(c0\ d1\ k + c1\ d0\ k + 2\,c0\ d0 + 2\,c1\ d1\right) a(n)\,b(n)$$
$$+\left(c0\ d1 - c1\ d0\right)^2 \left(c0\ c1\ k + c0^2 + c1^2\right) b(n)^2 \ = \ \left(c0\ d1 - c1\ d0\right)^4 \quad .$$

An important special case solves **Pell's Equation**.

Special Theorem (solving Pell's Equation) Let b and k be integers. Define two sequences of integers $A(n), B(n)$

$$\sum_{n=0}^{\infty} A(n)x^n \ = \ \frac{1-kx}{1-2kx+x^2} \quad ,$$

$$\sum_{n=0}^{\infty} B(n)x^n \ = \ \frac{bx}{1-2kx+x^2} \quad .$$

Let

$$P(X,Y) = X^2 - \left(\frac{k^2-1}{b^2}\right) Y^2 - 1 \quad ,$$

then

$$P(A(n),B(n)) \ = \ 0 \quad .$$

In particular, once we found the smallest non-trivial solution, (k,b), of **Pell's equation**, $k^2 - N\,b^2 \ = \ 1$, it enables us to get **all** solutions. Of course this is equivalent to the standard way, using expressions of the form $(k+\sqrt{N}b)^n$, and extracting the coefficients of $\sqrt{N}$, and 1, but our approach is simpler, and puts it in the context of solving diophantine equations within the C-finite ansatz.

We also have

Another General Theorem: Let $c_0, c_1,\ d_0, d_1$ and k be integers and define two sequences of integers $a(n), b(n)$

$$\sum_{n=0}^{\infty} a(n)x^n \ = \ \frac{c_0 + c_1 x}{1 - kx - x^2} \quad ,$$

$$\sum_{n=0}^{\infty} b(n)x^n \ = \ \frac{d_0 + d_1 x}{1 - kx - x^2} \quad ,$$

then there exists a homogeneous binary quadratic from $P(X, Y)$ with integer coefficients, and an integer C such that for all $n \geq 0$

$$P(a(n), b(n)) \ = \ C(-1)^n \quad .$$

Even more generally, we have the following theorem.

Very General Theorem: Let d be an integer larger than 1, and let $c_{i,j}$ $1 \leq i \leq d, 0 \leq j \leq d-1$ and $k_1, \dots, k_{d-1}$ be integers. Define d integer sequences $a_i(n)$ $(1 \leq i \leq d)$ by the generating functions

$$\sum_{n=0}^{\infty} a_i(n)x^n \ = \ \frac{c_{i,0} + \cdots + c_{i,d-1}x^{d-1}}{1 - k_1x - k_2x^2 - \dots k_{d-1}x^{d-1} + (-1)^d x^d} \quad , \quad (1 \leq i \leq d) \quad ,$$

then there exists a homogeneous polynomial $P(X_1, \dots, X_d)$, with integer coefficients, of degree d and an integer C such that

$$P(a_1(n), a_2(n), \dots, a_d(n)) \ = \ C \quad .$$

Analogously

Very General Theorem': Let d be an integer larger than 1, and let $c_{i,j}$ $1 \leq i \leq d, 0 \leq j \leq d-1$ and $k_1, \dots, k_{d-1}$ be integers. Define d integer sequences $a_i(n)$ $(1 \leq i \leq d)$ by the generating functions

$$\sum_{n=0}^{\infty} a_i(n)x^n \ = \ \frac{c_{i,0} + \cdots + c_{i,d-1}x^{d-1}}{1 - k_1x - k_2x^2 - \dots k_{d-1}x^{d-1} + (-1)^{d+1} x^d} \quad , (1 \leq i \leq d)$$

then there exists a homogeneous polynomial $P(X_1, \dots, X_d)$ of degree d (with integer coefficients) and an integer C such that

$$P(a_1(n), a_2(n), \dots, a_d(n)) \ = \ C(-1)^n \quad .$$

Back to the Quadratic Case

Using symbolic computations and experimental mathematics, we can get an alternative approach for *solving* quadratic diophantine equations. Of course, this is all classical, and modern treatments can be found, e.g. in the excellent texts [S] and [AA]. Our approach also uses continued fractions, but in a much more simple-minded way, without the usual human-generated infra-structure. This is implemented in the Maple package `Pell.txt`. The main procedure is `SolQuad(Q,m,n,t,K)` that inputs a quadratic binary form with integer coefficients, $Q(m,n)$, in the variables m and n and a symbol t and a positive integer K (a parameter for 'guessing' it can always be increased until success is reached), and outputs two rational functions, $f_1(t), f_2(t)$ (with identical, quadratic, monic denominators) that are the generating functions for the sequences $a(i), b(i)$ that satisfy $Q(a(i), b(i)) = constant$. From now on we can take it as a **black box**.

Eri Jabotinsky's trick

Ramanujan was not the first one to have a parametric solution to the equation $X^3+Y^3+Z^3+W^3 = 0$, this honor goes to Euler (see [D], vol. 2, Chap. XXI, pp. 552-553), but Ramanujan had quite a few of them, and probably found them all by himself. In a delightful article meant for teenagers, Eri Jabotinsky [J] explained how Ramanujan *might* have found them.

Suppose that you have two distinct solutions (x, y, z, w) and (x', y', z', w') of the equation $X^3 + Y^3 + Z^3 + W^3 = 0$, in other words

$$x^3 + y^3 + z^3 + w^3 = 0 \quad , \quad (x')^3 + (y')^3 + (z')^3 + (w')^3 = 0 \quad .$$

Let c and d be *undetermined coefficients* for now, and let's try to find a brand new solution of the form

$$X = cx + dx' \quad , \quad Y = cy + dy' \quad , \quad Z = cz + dz' \quad , \quad W = cw + dw' \quad .$$

Expanding $X^3 + Y^3 + Z^3 + W^3$ we get

$$c^3x^3 + 3c^2dx^2x' + 3cd^2xx'^2 + d^3x'^3 \quad +$$

$$c^3y^3 + 3c^2dy^2y' + 3cd^2yy'^2 + d^3y'^3 \quad +$$

$$c^3z^3 + 3c^2dz^2z' + 3cd^2zz'^2 + d^3z'^3 \quad +$$

$$c^3w^3 + 3c^2dw^2w' + 3cd^2ww'^2 + d^3w'^3 \quad .$$

Collecting terms, and equating to 0, we get

$$c^3(x^3 + y^3 + z^3 + w^3)$$

$$+3cd(c(x^2x' + y^2y' + z^2z' + w^2w') + d(xx'^2 + yy'^2 + zz'^2 + ww'^2))$$

$$+d^3(x'^3 + y'^3 + z'^3 + w'^3) \; = \; 0 \quad .$$

The first and last terms vanish by assumption, so we need to choose integers c and d such that

$$c(x^2x' + y^2y' + z^2z' + w^2w') + d(xx'^2 + yy'^2 + zz'^2 + ww'^2) \; = 0.$$

Taking

$$c = x'^2 x + y'^2 y + z'^2 z + w'^2 w \quad , \quad d = -(x^2 x' + y^2 y' + z^2 z' + w^2 w') \quad ,$$

would work, so out of the original two **numerical** special solutions (x, y, z, w) and (x', y', z', w') we get yet-another specific numerical solution. By pure inspection, (even by hand!) it is very fast to come up with specific numerical solutions, the most famous ones being $(3, 4, 5, -6)$ and $(9, 10, -1, -12)$ [of taxicab fame]. But in addition, there is a trivial 'doubly-infinite' solution $(m, -m, n, -n)$. By using the Eri Jabotinsky trick (that Jabotinsky conjectured was Ramanujan's way), 'morphing' any specific solution with the symbolic solution $(m, -m, n, -n)$, we get a doubly-infinite **non-trivial** solution consisting of quadratic polynomials in m and n.

Back to Mike Hirschhorn

Mike Hirshhorn's starting point (and presumably Ramanujan's) was the identity

$$(A^2 + 7AB - 9B^2)^3 + (2A^2 - 4AB + 12B^2)^3 + (-2A^2 - 10B^2)^3 + (-A^2 + 9AB + B^2)^3 = 0 \quad . \quad (2)$$

Using human ingenuity, he then solved the diophantine equation $-A^2 + 9AB + B^2 = \pm 1$, by [essentially] finding two rational functions whose respective Taylor coefficients satisfy this equation. Then using further *manipulatorics*, he found generating functions for the other quantities, thereby rediscovering, *ab initio*, Ramanujan's theorem. All this can be streamlined in the context of the C-finite ansatz, and has been implemented in the Maple package `RamanujanCubic.txt`, mentioned above. But what is the point? Who cares about a computerized redux of the beautiful human-generated work of Hirschhorn, Jabotinsky (and Ramanujan).

The point is that we can **generalize**. Eri Jabotinsky's trick works just as well for the more general cubic diophantine equation

$$a\,X^3 + a\,Y^3 + b\,Z^3 + b\,W^3 = 0,$$

for *any* integers a and b. It is easy, by pure brute force, to come up with specific numerical solutions (even by hand). The symbolic solution $(m, -m, n, -n)$ is still a solution. Using the Jabotinsky trick, we get four quadratic polynomials $P_1(m, n), P_2(m, n), P_3(m, n), P_4(m, n)$ such that

$$a\,P_1(m, n)^3 + a\,P_2(m, n)^3 + b\,P_3(m, n)^3 + bP_4(m, n)^3 = 0 \quad .$$

Then we can use (our version of) the well-known algorithm (implemented in our Maple package) to find a pair of rational functions $f_1(t), f_2(t)$ whose respective Taylor coefficients solve $P_4(m, n) = 1$ (or failing this [there is not always a solution]) $P_4(m, n) = c$ for some fixed small integer c. Then using the 'C-finite calculator' [Z2], also built-in into the package, one gets as many Ramanujan-like theorems solving diophantine equations of the form $a\,X^3 + a\,Y^3 + b\,Z^3 = c$, (with a different c, of course) for any desired a and b, and for some emerging constant c, like the sample theorem above.

For 640 additional such theorems, see the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oRamanujanCubic1.txt` .

Of course, using the Maple package

`http://www.math.rutgers.edu/~zeilberg/tokhniot/RamanujanCubic.txt` ,

readers can generate many more such theorems.

Morals

1. Read Children books, especially such delightful ones as Amy Alznauer's about Ramanujan.

2. Reread your favorite math articles from when you were a teenager, in this case made possible by Gadi Aleksandrowicz.

3. Reread insightful papers, like [Ha1] trying to do a **human deconstruction** of geniuses like Ramanujan.

4. Teach these tricks to a computer, doing **computerized deconstruction** (see e.g. [Z1])

5. By doing (usually) minor tweaking to the program, generalize it, and generate as many Ramanujan-like theorems as you wish.

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[1] Eri Jabotinsky and Theodore Motzkin were both sons of Zionist leaders, Ze'ev Jabotinsky [the leader of the militant branch that inspired the Jewish terrorist groups Etzel and Lechi], and Leo Motzkin, a close friend of Theodore Herzl. Other examples are Herb Wilf, whose father Alexander lead the fund-raising drive for the *irgun* before 1948, and Gil Kalai, whose father, Hanoch, was second-in-command in Yair Stern's organization, *Lechi*.