Multi-Variable Zeilberger and Almkvist-Zeilberger Algorithms
and the
Sharpening of Wilf-Zeilberger Theory

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q-Notation. For k integer, $[a]_k := (1-q^a)(1-q^{a+1})\cdots(1-q^{a+k-1})$, if $k \geq 0$ and $[a]_k := 1/[a+k]_{-k}$ if $k < 0$. In order to avoid too many subscripts, we will denote $[a]_k$ by $qRF(a\,,\,k)$.

A Multi-Variate q-Zeilberger Algorithm

q-Theorem. Let

$$F(n;k_1,\ldots,k_r) = POL(n;k_1,\ldots,k_r)\cdot H(n;k_1,\ldots,k_r) \quad , \qquad (qMultiProperHypergeometric)$$

where $POL(n;k1,\ldots,k_r)$ is a Laurent polynomial in $(q^n, q^{k_1},\ldots,q^{k_r})$, and

$$H(n;k_1,\ldots,k_r) = \frac{\prod_{j=1}^{A} qRF(a''_j\,,\,a'_j n + \sum_{i=1}^{r} a_{ji}k_i)}{\prod_{j=1}^{C} qRF(c''_j\,,\,c'_j n + \sum_{i=1}^{r} c_{ji}k_i)} \cdot q^{Q(n;k_1,\ldots,k_r)} \cdot \prod_{i=1}^{r} z_i^{k_i} \quad ,$$
$$(qMultiPureHypergeometric)$$

where the a'_j, c'_j are *non-negative* integers and the a_{ji}, c_{ji} are *integers*, while a''_j, c''_j and $z_1,\ldots,z_r$ are *commuting indeterminates*, and $Q(n;k_1,\ldots,k_r)$ is a quadratic form in $(n,k_1,\ldots,k_r)$. Then there exists an integer L, to be *explicitly* constructed in the course of the proof, polynomials in q^n, $e_0(q^n), e_1(q^n),\ldots,e_L(q^n)$, *not all zero*, and r rational functions of $(q^n,q^{k_1},\ldots,q^{k_r})$, $R_I(n;k_1,\ldots,k_r)$ $(I=1,\ldots,r)$ such that

$$G_I(n;k_1,\ldots,k_r) := R_I(n;k_1,\ldots,k_r)F(n;k_1,\ldots,k_r)$$

satisfy

$$\sum_{i=0}^{L} e_i(q^n)F(n+i,k) = \sum_{I=0}^{r}[G_I(n;k_1,\ldots,k_{I-1},k_I+1,k_{I+1},\ldots,k_r) - G_I(n;k_1,\ldots,k_r)] \quad .$$
$$(qZtuple)$$

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