von Neumann and Newman Pokers with Finite Decks

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Abstract

John von Neumann studied a simplified version of poker where the "deck" consists of *infinitely* many cards, in fact, all real numbers between **0** and **1**. We harness the power of computation, both numeric and symbolic, to investigate analogs with *finitely* many cards. We also study finite analogs of a simplified poker introduced by D.J. Newman, and conclude with a thorough investigation, fully implemented in Maple, of the three-player game, doing *both* the finite and the infinite versions. This paper is accompanied by two Maple packages and numerous output files; however, no knowledge of Maple is needed, as all relevant information is provided within the paper.

Keywords: Poker, Two-player game, Three-player game, Game theory

1 Prelude

Welcome to the world of poker, where strategy and probability rule. Picture yourself at the poker table, every decision a crucial step toward victory or defeat. Poker has intrigued mathematicians for decades as a window into decision-making and game theory. Pioneers like Émile Borel, John von Neumann, Harold W. Kuhn, John Nash, and Lloyd Shapley ([1],[8],[4],[6]) who believed that real-life scenarios mirror poker with their elements of bluffing and strategic thinking, have simplified the complexities of the game, making it tractable for game theoretic analysis.

Quick Refresher: Game Theory

In game theory, a *game* refers to any situation where players make decisions that result in outcomes based on the choices of all involved. A *strategy* is a complete plan of action a player will follow in various situations throughout the game. A *pure strategy* is a strategy in which a player makes a specific choice or takes a specific action with certainty whenever a particular situation arises in the game. When players use a *mixed strategy*, they randomize over possible moves, assigning a probability to each move, instead of choosing a single, deterministic action. A *Nash Equilibrium (NE)* occurs when no player can benefit from changing their strategy while the other players keep theirs unchanged. A *mixed Nash Equilibrium* is a type of NE where at least one player uses a mixed strategy, ensuring no player can improve their payoff by individually changing their strategy.

von Neumann Poker

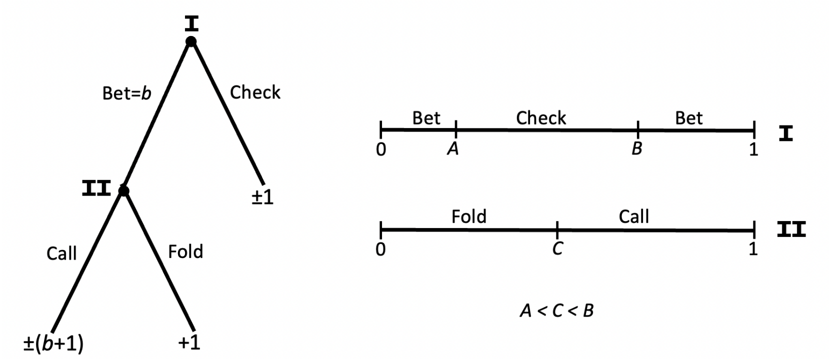

Fig. 1 The betting tree and Nash equilibrium strategies for von Neumann Poker

In the original version [8], von Neumann proposed, and solved, the following game of poker with an uncountably infinite deck, namely all the real numbers between 0 and 1. Fix a bet size, b. Player I and Player II are dealt (uniformly at random) two "cards", real numbers x and y, in the interval $[0, 1]$. They each see their own card, but have no clue about the opponent's card. At the start they each put one dollar into the pot (the so called *ante*), so now the pot has two dollars.

Figure 1 illustrates the "betting tree" of this game. Here, Player I looks at his card, and decides whether to *check*, in which case each of the players shows their cards, and whoever has the largest card wins the pot. On the other hand he has an option to *bet*, putting b additional dollars in the pot. Now the game turns to Player II. She can decide to *fold*, in which case player I gets the pot, resulting in a gain of 1 dollar for Player I, (and a loss of 1 dollar for player II), or be **brave** and *call*, putting her own b dollars into the pot, that now has $2b+2$ dollars. The cards are compared in a *showdown* and whoever has the larger card, wins the whole pot, resulting in a gain of $b+1$ dollars for the winner, and a loss of $b+1$ for the loser.

von Neumann proved that the following pair of strategies is a pure *Nash Equilibrium*, i.e. if the players both follow their chosen strategy, neither of them can do better (on average) by doing a different strategy.

The von Neumann advice

von Neumann identified the cuts A, B and C in the right panel of Figure 1, and proposed the following strategies.

- Player I: If $0 < x < \frac{b}{(b+4)(b+1)}$ or $\frac{b^2+4b+2}{(b+4)(b+1)} < x < 1$ you should **bet**, otherwise **check**.
- Player II: If $0 < y < \frac{b(b+3)}{(b+4)(b+1)}$ you should **fold**, otherwise **call**.

Note that Player II's strategy corresponds to *honest common sense*, there is some *cut-off* that below it you should be conservative, and "cut your losses" giving up the one dollar, and not risking losing b additional dollars, and above it, be brave, and *go for it*.

Now an *honest common sense* would tell you that Player I would also have his own cutoff, check if his card is below it, and bet if it exceeds it. But this is **not** optimal. If Player I has a low card, he should **bluff**, and 'pretend' that he has a high card, and player II would be intimidated into folding.

Sad but true, "honesty is **not** the best policy". Indeed the game favors Player I, and his expected gain is $\frac{b}{(b+4)(b+1)}$.

When $b=2$, the advice spells out as follows:

- Player I: if $0 < x < \frac{1}{9}$ or $\frac{7}{9} < x < 1$ you should **bet**, otherwise **check**.
- Player II: If $0 < y < \frac{5}{9}$ you should **fold**, otherwise **call**.

The expected value, i.e. the value of the game (for Player I) is $\frac{1}{9}$. It can be shown that $b=2$ maximizes Player I's payoff under the Nash equilibrium strategies.

Finitely Many Cards

What we *don't* like about the original von Neumann version is that the deck is infinite. In **real life** there are only *finitely* many cards, and in fact, not that many. We were wondering whether there exists pure Nash equilibria when there are only finitely many cards. Before we dive into the world of finite poker, interested readers can find detailed implementations of this work in the Maple package: https://sites.math.rutgers.edu/~zeilberg/tokhniot/FinitePoker.txt, along with an expanded version of the current work from a computational perspective in [3].

Finding all pure NEs via von Neumann's Minimax Theorem

Let $n \geq 2$ be a fixed positive integer representing the number of cards in the deck, numbered $1, 2, \ldots, n$. Additionally, let $b \geq 1$ be a fixed positive integer denoting the *bet size*. In this section, we aim to identify the set of **all** pure Nash equilibria for a given pair (n, b), which, as we will see, may occasionally be empty (hereafter, we use NE to refer to Nash Equilibrium).

As we did not make *any* assumptions about 'plausible' strategies, *a priori*, a strategy for Player I can be *any* subset, S_1, of $\{1, \ldots, n\}$, that advises: 'If your card belongs to S_1 you should **bet**, otherwise, **check**'. Similarly a strategy for Player II, S_2, can be any such subset, that tells her to call iff her card $j \in S_2$. We then construct the *paytable*, which is a $2^n \times 2^n$ payoff matrix. The smallest payoff matrix of size 4×4 for a bet size $b = 2$ and $n = 2$ cards is given in Figure 2.

To solve for a pure Nash equilibrium in our finite poker game, it is fitting to revisit *John von Neumann's* minimax theorem, first published in 1928 [7]. This theorem remains a cornerstone of game theory to this day, and it is a privilege to apply his celebrated result to solve the finite version of his poker model.

In the context of a two-player zero-sum game, given a payoff matrix, the theorem states that if the *row maximin* equals the *column minimax*, a *pure Nash equilibrium* (or saddle point) is guaranteed to exist. In particular, at an equilibrium strategy pair $[S_1, S_2]$, Player I (the row player) aims to maximize their worst possible payoff, while Player II (the column player) seeks to minimize Player I's best payoff. The *value of the game* (the expected gain of Player I and the corresponding loss of Player II) is the outcome at this equilibrium pair, representing the optimal result for both players.

However, if these values do not coincide, the equilibrium will typically involve mixed strategies, where both players randomize their decisions. A historical note related to this: Von Neumann and Morgenstern showed that, for a zero-sum game, there must always exist at least one mixed Nash equilibrium, as demonstrated in their 1944 book *Theory of Games and Economic Behavior* [8]. In 1950, John Nash generalized this concept in his paper *Non-Cooperative Games* [5] to the non-zero-sum game. We will talk about mixed NEs later on. But for now, let's explore the pure ones.

Referring back to Figure 2, for the payoff matrix when the bet size is $b = 2$ and we have only 2 cards, the value of the row maximin equals the column minimax, both being 0, at two pairs $[S_1, S_2]$ of pure Nash equilibria:

$$[\{\}, \{2\}] \quad \text{and} \quad [\{2\}, \{2\}].$$

In both of them Player II calls if her card is 2 and folds if her card is 1, while Player I always checks in the first strategy, and checks if his card is 1 in the second strategy. This is not very interesting, since the value of the the game is 0.

Fixing the bet size to 2 and setting $n = 3$ cards (so the payoff matrix is now 8×8) is, frankly, a bit dull, as it leads to just two trivial pairs of pure NEs: $[\phi, \{3\}]$ and $[\{3\}, \{3\}]$. Increasing the number of cards to 4, 5, or even 6 (while the size of the payoff matrix grows exponentially) doesn't add much excitement—they're all empty, as the

Strategy	$S_2 = \{\ \}$ Always Fold	$S_2 = \{1\}$ Call if "1", Fold if "2"	$S_2 = \{2\}$ Call if "2", Fold if "1"	$S_2 = \{1,2\}$ Always Call	Row Min
$S_1 = \{\ \}$ Always Check	(-1+1)/2 = 0	(-1+1)/2 = 0	(-1+1)/2 = 0	(-1+1)/2 = 0	0
$S_1 = \{1\}$ Bet if "1", Check if "2"	(+1+1)/2 = 1	(+1+1)/2 = 1	(-3+1)/2 = -1	(-3+1)/2 = -1	-1
$S_1 = \{2\}$ Bet if "2", Check if "1"	(-1+1)/2 = 0	(-1+3)/2 = 1	(-1+1)/2 = 0	(-1+3)/2 = 1	0
$S_1 = \{1,2\}$ Always Bet	1	(+1+3)/2 = 2	(-3+1)/2 = -1	(-3+3)/2 = 0	-1
Column Max	1	2	0	1	

Fig. 2 Payoff matrix for $n = 2$ and $b = 2$, along with the values of the row minima and column maxima

value of the row maximin does not equal the column minimax, resulting in no pure NEs.

But now comes a nice surprise, with 7 cards, we get not one, not two, but *three pure, non-trivial* Nash equilibria! In all of them, Player I bets if their card is in $\{1, 6, 7\}$, while Player II calls if her card belongs to any of the following sets: $\{3, 6, 7\}$, $\{4, 6, 7\}$, or $\{5, 6, 7\}$. The value of the game is $\frac{2}{21}$.

So with 7 cards we already have bluffing! If Player I has the card labeled 1, he should bet even though he would definitely lose the bet if Player II calls.

Moving right along, with 8 cards, we also get three pure NEs. For all of them Player I bets iff his card belongs to $\{1, 7, 8\}$, but Player II calls if her card is in either $\{4, 7, 8\}$, $\{5, 7, 8\}$, or $\{6, 7, 8\}$. The value of the game is $\frac{3}{28}$, getting tantalizingly close to von Neumann's $\frac{1}{9}$.

Since the sizes of the payoff matrices grow exponentially, and we did not make *any plausibility assumptions*, there is only so far we can go with this naive *vanilla* approach. But nine cards are still doable. Indeed there are seven pure NEs in this case. For all of them $S_1 = \{1, 8, 9\}$, but Player II has seven choices, all with four members, including, of course, $\{6, 7, 8, 9\}$.

To overcome the *exponential explosion*, we can stipulate that Player I's strategy **must** be of the form:

"Check iff $i \in \{A, A + 1, \ldots, B\}$ for some $1 \le A < B \le n$, "

while Player II's must be of the form:

"Call iff $j \in \{C, C + 1, \ldots, n\}$ for some $1 \le C \le n$."

Now we can go much further, which leads to a nice result: *If n is a multiple of 9 then the (restricted) pure NEs are as expected, namely the value of the game is $\frac{1}{9}$ and the strategy for player I is: check if $\frac{1}{9}n < i \le \frac{7}{9}n$, bet otherwise and for Player II: call iff $j > \frac{5}{9}n$.*

If n is not a multiple of 9, then the values are close, but a little less. For example for $n = 26$ the value is $\frac{36}{325} = 0.110769$. For $n = 25$, the value is $\frac{11}{100} = 0.11$.

2 Mixed NEs via Linear Programming

The study of mixed strategies in two-person zero-sum games can be elegantly formulated as a primal-dual linear programming (LP) problem. A *mixed* strategy involves each player choosing optimal actions according to a probability distribution, introducing uncertainty. An equilibrium solution to this dual pair of linear programs reveals optimal mixed strategies (mixed NE) for both players.

Slow LPs for mixed NE

Recall our scenario: the pot starts at 1+1, with only Player I able to bet a fixed amount b. Given the 2^n by 2^n payoff matrix (m_{ij}) as input, Player I aims to maximize his worst-case expected gain, minimizing over all possible actions of Player II. This objective is framed as an LP by introducing variable v_1 to represent this minimum, ensuring Player I's expected gain is at least v_1 for every action of Player II, and maximizing v_1. Similarly, from Player II's viewpoint, the goal is to minimize her worst-case expected loss, maximizing over all actions of Player I. This involves introducing variable v_2 to represent this maximum, and setting the objective to minimize v_2.

To formulate the primal-dual LP, let $\mathbf{x} = (x_1, \ldots, x_{2^n})$ be the mixed strategy probability of Player I to maximize v_1. Let $\mathbf{y} = (y_1, \ldots, y_{2^n})$ be the mixed strategy probability of Player II to minimize v_2.

Primal: Maximize v_1

$$\text{s.t. } \sum_{i=1}^{2^n} x_i \cdot m_{ij} \geq v_1 \quad \text{for } j = 1, ..., 2^n$$
$$\sum_{i=1}^{2^n} x_i = 1$$
$$x_i \geq 0 \quad \text{for } i = 1, ..., 2^n.$$

Dual: Minimize v_2

$$\text{s.t. } \sum_{j=1}^{2^n} m_{ij} \cdot y_j \leq v_2 \quad \text{for } i = 1, ..., 2^n$$
$$\sum_{j=1}^{2^n} y_j = 1$$
$$y_j \geq 0 \quad \text{for } j = 1, ..., 2^n.$$

By the minimax theorem at an equilibrium, $v_1 = v_2 = v^*$, which represents the value of the game.

For example, with $n = 4$ cards and a bet size $b = 2$, one can set up a 16×16 payoff matrix and solve above LPs for a mixed strategy NE, resulting in the following:

- Player I has two strategies: (1.1) with probability 1/2, bet if his card is 4 and fold if his card is 1, 2, or 3; and (1.2) with probability 1/2, bet if his card is 1 or 4, and fold if his cards are 2 or 3.
- Player II has two strategies: (2.1) with probability 1/2, call if her card is 4 and fold if her card is 1, 2, or 3; and (2.2) with probability 1/2, call if her cards are 2 or 4, and fold if her cards are 1 or 3.
- The value of the game is $\frac{1}{12}$.

However, due to the exponentially large size of the matrix, practical limitations arise, preventing us from considering more than 6-7 cards without the inconvenience of reducing the dominated rows and columns of the payoff matrix.

Fast LPs for mixed NE

The NE can be considered from a different perspective, where focusing on the specific card each player receives reduces the number of constraints from exponential to linear.

A strategy for Player I is given by a vector $\mathcal{P} = [p_1, \ldots, p_n]$ that tells him: if his card is i, bet with probability p_i, and check with probability $1 - p_i$.

A strategy for Player II is given by a vector $\mathcal{Q} = [q_1, \ldots, q_n]$ that tells her: if her card is j, call with probability q_j, and fold with probability $1 - q_j$.

Before we discuss the Fast LP formulation, let's mention that given card-by-card strategies, $\mathcal{P}$ and $\mathcal{Q}$, it is easy to compute the *expected payoff* (for Player I), as a bilinear form in the p_i's and q_j's:

$$\texttt{Payoff}(n, b, \mathcal{P}, \mathcal{Q}) = \frac{1}{n(n-1)} \Bigg(\sum_{i=1}^{n} \sum_{j=1}^{i-1} (1 - p_i) - \sum_{i=1}^{n} \sum_{j=i+1}^{n} (1 - p_i) + \sum_{i=1}^{n} \sum_{j=1}^{i-1} p_i (1 - q_j) + \sum_{i=1}^{n} \sum_{j=i+1}^{n} p_i (1 - q_j) + (b+1) \sum_{i=1}^{n} \sum_{j=1}^{i-1} p_i q_j - (b+1) \sum_{i=1}^{n} \sum_{j=i+1}^{n} p_i q_j \Bigg).$$

Let's now get back to the Fast LP for Player I, which contains two sets of constraints. Each set corresponds to the expected payoff (over distribution $\mathcal{P}$), conditioned on the card that Player II has and whether she calls or folds:

$$\begin{aligned} &\text{Maximize } \frac{1}{n} \sum_{j=1}^{n} v_j \\ \text{s.t. } &\frac{1}{n-1} \sum_{i \neq j} \left(\texttt{Call}(i, j, b+1) \cdot p_i + \texttt{Call}(i, j, 1) \cdot (1 - p_i) \right) \geq v_j \quad j = 1, \ldots, n \text{ (Player II calls)} \\ &\frac{1}{n-1} \sum_{i \neq j} \left(p_i + \texttt{Call}(i, j, 1) \cdot (1 - p_i) \right) \geq v_j \quad j = 1, \ldots, n \text{ (Player II folds)} \\ &0 \leq p_i \leq 1 \quad i = 1, \ldots, n, \end{aligned} \tag{VN-I}$$

where the procedure $\texttt{Call}(i, j, R)$ is defined based on whether the card i is larger than card j or not:

$$\texttt{Call}(i, j, R) = \begin{cases} R & \text{if } i > j \\ -R & \text{if } i < j. \end{cases}$$

Similarly, for the Fast LP for Player II, the constraints are calculated based on the expected loss (over distribution $\mathcal{Q}$), conditioned on the card that Player I has and whether he bets or checks:

$$\begin{aligned}
&\text{Minimize } \frac{1}{n}\sum_{i=1}^{n} v_i \\
\text{s.t. } &\frac{1}{n-1}\sum_{j\neq i}\left(\texttt{Call}(i,j,b+1)\cdot q_j + (1-q_j)\right) \le v_i \quad i=1,\ldots,n \text{ (Player I bets)} \\
&\frac{1}{n-1}\sum_{j\neq i}\texttt{Call}(i,j,1) \le v_i \quad i=1,\ldots,n \text{ (Player I checks)} \\
&0 \le q_j \le 1 \quad j=1,\ldots,n.
\end{aligned} \tag{VN-II}$$

Now things get interesting much sooner. Even with just three cards, we already have bluffing! With a bet size of 1 (note the difference in bet size from the usual $b = 2$), the results are as follows:

- Player I's strategy is: If your card is 1, bet with probability $\frac{1}{3}$ and check with probability $\frac{2}{3}$. If your card is 2 then **definitely check**, while if your card is 3 then you should **definitely bet**.
- Player II's strategy is: If your card is 1, **definitely fold**, if your card is 2, call with probability $\frac{1}{3}$ and fold with probability $\frac{2}{3}$, while if your card is 3 then **definitely call**.
- The value of the game is $\frac{1}{18} \approx 0.055555\ldots$.

So already with three cards, Player I should sometimes bluff if his card is 1, but only with probability $\frac{1}{3}$.

Note that a pure Nash equilibrium is also a mixed one, and indeed, in some cases, we obtain pure Nash equilibria. For example, with $n = 18$ cards and $b = 2$, the result is as follows:

- Player I: Bet iff your card is in $\{1, 2, 15, 16, 17, 18\}$.
- Player II: Call iff your card is in $\{11, \ldots, 18\}$.
- The value of the game is $\frac{1}{9} \approx 0.111111111\ldots$.

Beyond the results obtained for the more realistic scenario of finitely many cards and their computational efficiency of the Fast LP model, our findings provide crucial insights into the continuous case, shedding light on why Player I's Nash equilibrium strategy in von Neumann's poker follows the pattern depicted in Figure 1: bluffing when his card is small.

3 DJ Newman Poker

Not as famous as John von Neumann, but at least as brilliant, is Donald J. Newman, the third person to be Putnam fellow in three consecutive years. He was a good friend of John Nash. In a fascinating four-page paper [9] in *Operations Research*, he proposed his own version of poker, where the bet size is **not** fixed, but can be decided by Player I, including betting 0, that is the same as *checking*.

In his own words (now the players are A and B) :

A and B each ante 1 *dollar and are each dealt a 'hand,' namely a randomly chosen real number in* $(0, 1)$. *Each sees his, but not the opponent's hand. A bets* *any amount he chooses (≥ 0); B 'sees' him (i.e. calls, betting the same amount) or folds. The payoff is as usual.*

The DJ Newman advice

The betting tree and Nash equilibrium strategies are the same as those in von Neumann Poker, as shown in Figure 1. However, Player I is allowed to bet with different positive amounts.

- **Player I**

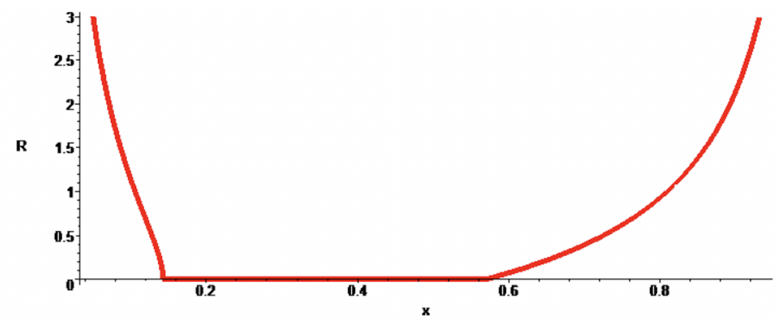

Fig. 3 Relation between the value of the card x and the optimal bet size R for Player I

Case 1: For a given card $x < A = \frac{1}{7}$, Player I should bet an amount R that satisfies the relation:

$$x = \frac{1}{7} - \frac{R^2(R+6)}{7(R+2)^3} = \frac{4(3R+2)}{7(R+2)^3}.$$

Case 2: For a given card $\frac{1}{7} = A < x < B = \frac{4}{7}$, Player I should check.

Case 3: For a given card $x > B = \frac{4}{7}$, Player I should bet an amount R that satisfies the relation:

$$x = \frac{R^2 + 4R + 2(1/7) + 2}{(R+2)^2} = \frac{7R^2 + 28R + 16}{7(R+2)^2}.$$

- **Player II**

In response to a bet amount $R > 0$ from player I, player II should fold if her card $y < C$ and call if $y > C$, where

$$C = \frac{R + 2(1/7)}{R+2} = \frac{7R+2}{7R+14}.$$

The value of the game (for Player I) is $\frac{1}{7}$. (This is higher than $\frac{1}{9}$ in von Neumann's game, as Player I has more freedom to bet.) The detailed calculation of this result can be found in Newman's original paper [9], or check our supplementary material at https://thotsaporn.com/SupplementN.pdf.

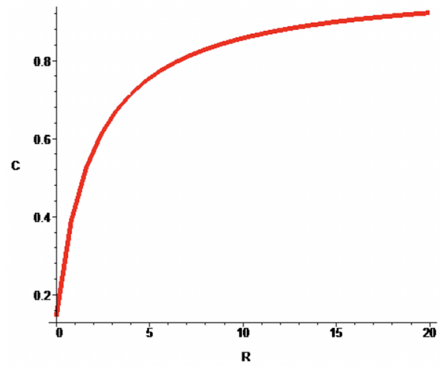

Fig. 4 Relation between the amount of bet R and the value of the card C that is sufficient to call the bet for Player II

Finitely Many Cards

But in real life, there is always a finite number of cards, and no one can bet arbitrarily large amounts. Once again, we focus on the finite deck version, which is set up as follows: The inputs are integers $n \geq 2$ and $b \geq 1$, where each player is dealt a different card from $\{1, \ldots, n\}$, and Player I's decision, upon seeing his card i, is to choose an amount s from $\{0, \ldots, b\}$ to bet, where $s = 0$ corresponds to checking.

In this game the number of strategies are even larger, and we will not bother with the 'vanilla' approach to find pure NEs. Instead, we will look for (Fast LP) mixed strategies right away.

Player I's payoff maximization

Player I's strategy space consists of $n \times (b+1)$ matrix, $(p_i[s])$, where $p_i[s]$ $(1 \leq i \leq n, 0 \leq s \leq b)$ is the probability that if he has card i, he would bet s dollars (of course, the row-sums should add-up to 1). The LP formulation is analogous to that of (VN-I) in the previous section. Let's point out the differences to gain some insights. Recall that each constraint corresponds to the card that Player II has and her choice of action. In (VN-I), Player II can either call or fold, and she can have one of the n cards. Hence, there are a total of $2n$ constraints.

In our current scenario, however, Player II's decision depends on both her card and Player I's proposed bet amount s. Let $S_b := \{0, \ldots, b\}$. We define $\mathcal{P}(S_b)$ as the set containing all possible strategies of Player II regarding whether to call or fold. That is, each $Y \in \mathcal{P}(S_b)$ represents a strategy where Player II will call if $s \in Y$. Therefore, for a fixed card j and strategy $Y \in \mathcal{P}(S_b)$ of Player II, the constraint is: *"Player II calls if she holds card j and the proposed bet amount $s \in Y$; otherwise, she folds."* The total number of these constraints amounts to $n \cdot 2^b$.

For example, when $b = 4$,

$$\mathcal{P}(S_4) = \{\{0\}, \{0,1\}, \{0,2\}, \{0,3\}, \{0,4\}, \{0,1,2\}, \{0,1,3\}, \{0,1,4\}, \{0,2,3\}, \{0,2,4\} \\ \{0,3,4\}, \{0,1,2,3\}, \{0,1,2,4\}, \{0,1,3,4\}, \{0,2,3,4\}, \{0,1,2,3,4\}\}.$$

With this setup, we derive the following LP:

$$
\begin{aligned}
&\text{Maximize } \frac{1}{n}\sum_{j=1}^{n} v_j \\
\text{s.t. } &\frac{1}{n-1}\sum_{i\neq j}\left(\sum_{s\in Y}\texttt{Call}(i,j,s+1)\cdot p_i[s] + \sum_{s\in(S_b\setminus Y)} p_i[s]\right) \geq v_j \quad \underbrace{j=1,\ldots,n;\ Y\in\mathcal{P}(S_b)}_{\text{total } n\cdot 2^b \text{ constraints}} \\
&\sum_{s=0}^{b} p_i[s] = 1, \quad i=1,\ldots,n \\
&p_i[s]\geq 0, \quad s=0,\ldots,b;\ i=1,\ldots,n
\end{aligned}
\tag{DJN-I}
$$

Player II's loss minimization

While Player's II's strategy is also an $n\times(b+1)$ matrix, formulating the LP is much simpler. Let's denote the matrix by $(q_j[s])$ where $q_j[s]$ is the probability of calling if her card is j and the bet proposed by Player I is s (and as usual $1-q_j[s]$ is the corresponding probability of folding). In this case, there are a total of $n(b+1)$ constraints (not exponential as in the case of Player I). Also, the LP formulation straightforwardly extends from (VN-II):

$$
\begin{aligned}
&\text{Minimize } \frac{1}{n}\sum_{i=1}^{n} v_i \\
\text{s.t. } &\frac{1}{n-1}\sum_{j\neq i}\left(\texttt{Call}(i,j,s+1)\cdot q_j[s] + (1-q_j[s])\right) \leq v_i \quad \underbrace{s=0,\ldots,b;\ i=1,\ldots,n}_{\text{total } n(b+1) \text{ constraints}} \\
&q_j[0]=1 \quad j=1,\ldots,n \\
&0\leq q_j[s]\leq 1 \quad s=0,\ldots,b;\ j=1,\ldots,n.
\end{aligned}
\tag{DJN-II}
$$

For example, for the game with $n=7$ cards and the maximum bet size $b=3$, the value of this game is $\frac{13}{105}=0.1238095238$, which is still less than $\frac{1}{7}=0.1428571429$ (due to the limitation of the maximum bet amount). Optimal strategies for both players are given by the following matrices:

$$
\left(p_i^*[s]\right)_{7\times 4} = \begin{bmatrix} \frac{1}{15} & \frac{1}{3} & 0 & \frac{3}{5} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \left(q_j^*[s]\right)_{7\times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & \frac{1}{10} & 0 \\ 1 & 1 & 1 & \frac{2}{5} \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.
$$

We will now interpret the optimal strategies of Players I and II through the obtained $p_i^*[s]$ and $q_j^*[s]$. For Player I, if he holds card 1, his optimal bet amounts will be 0, 1, 2, and 3, with corresponding probabilities of $\frac{1}{15},\frac{1}{3},0,\frac{3}{5}$, respectively. If his

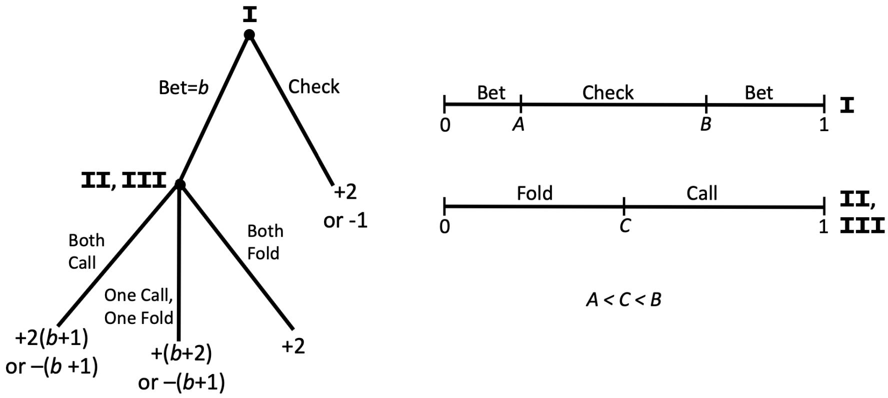

Fig. 5 Three-player poker. Left: The betting tree. Right: Conjectured Nash equilibrium strategies for a continuous deck.

card falls between 2 and 5, he will always check. If his card is 6, he will bet 1 with certainty, and if his card is 7, he will place the maximum bet of 3.

On the other hand, to interpret Player II's strategy, we examine the matrix $q_j^*[s]$ column by column, corresponding to the proposed bet amounts by Player I. In response to Player I's actions: if Player I checks (i.e., proposes a bet amount of 0), Player II will always call, which corresponds to ones in the first column. If Player I bets 1 (as in column 2), Player II will fold if her card is less than 4 and call otherwise. A similar interpretation can be made for the remaining columns.

We noticed that for any given n, there exists a maximal bet size after which the game has the same value. As n grows larger, and b reaches its saturation value, the value of the game seems to converge to the DJ Newman 'continuous' value $\frac{1}{7}$.

4 Three-player Poker Game

As early as 1950, future Economics Nobelists, John Nash and Lloyd Shapley [6], pioneered the analysis of a three-player poker game. They explored a simplified version where the deck contains only two kinds of cards, High and Low, in equal numbers. However, today, eighty years after von Neumann's analysis of poker, the dynamics of the three-player game therein remain unexplored. We now take the opportunity to analyze these dynamics in both their finite and infinite versions.

Finite deck

The three players each put 1 dollar into the pot. Player I acts first, choosing either to check or to bet a fixed integer amount $b > 0$. If Player I checks, the three hands are immediately compared, and the player with the highest hand wins the pot. However, if Player I bets, Players II and III have two choices: call or fold. The reader is invited to refer to the left panel of Figure 5, which depicts the betting tree for three players. (The right panel shows the conjectured Nash equilibrium strategies to be used in the next section for the continuous version of the game.)

Assume we are given three-dimensional payoff matrices $(M_l, l = 1, 2, 3)$ for the three players:

$$M_l = \left(m^l_{ijk}\right),$$

where $i, j, k = 1, 2, \ldots, 2^n$.

While its counterpart two-player game can be solved using linear programming, here we require nonlinear programming (NLP) [2]. The NLP formulation for the three-player game closely follows the LP model for the two players discussed in the previous section. Each player aims to minimize their expected loss, or the expected gain of the other players. For instance, given Player I's payoff matrix M_1, the other two players attempt to minimize the maximum potential loss incurred due to Player I's choices. This involves constraints that utilize matrix M_1 and the probability distributions $\mathbf{y} = (y_1, \ldots, y_{2^n})$ and $\mathbf{z} = (z_1, \ldots, z_{2^n})$ of Players II and III. These are embedded in the first set of constraints in the NLP formulation, which we will now formulate.

The Slow NLP for three players is given by:

$$\begin{aligned}
&\text{Minimize } \sum_{l=1}^{3} v^l \\
\text{s.t. } &\sum_{j,k=1}^{2^n} m^1_{ijk} \cdot y_j \cdot z_k \le v^1 \quad \text{for } i = 1, 2, ..., 2^n \\
&\sum_{i,k=1}^{2^n} m^2_{ijk} \cdot x_i \cdot z_k \le v^2 \quad \text{for } j = 1, 2, ..., 2^n \\
&\sum_{i,j=1}^{2^n} m^3_{ijk} \cdot x_i \cdot y_j \le v^3 \quad \text{for } k = 1, 2, ..., 2^n \\
&\sum_{i=1}^{2^n} x_i = 1, \qquad \sum_{j=1}^{2^n} y_j = 1, \qquad \sum_{k=1}^{2^n} z_k = 1 \\
&x_i, y_j, z_k \ge 0 \quad \text{for } i, j, k = 1, 2, ..., 2^n.
\end{aligned}$$

Note that if there are only two players, z_k in the above NLP formulation disappears, and the constraint functions become linear in the variables x_i and y_j. Thus, the problem can be decomposed into two separate LP (primal-dual) problems, as discussed earlier.

We now shift our focus to the Fast NLP formulation for three players, which aligns with the Fast LP formulation for two players, considering on the card each player receives. Recall a strategy for Player I is given by a vector $\mathcal{P} = [p_1, \ldots, p_n]$, indicating that if his card is i, he bets with probability p_i, and checks with probability $1 - p_i$. A strategy for Player II is given by a vector $\mathcal{Q} = [q_1, \ldots, q_n]$, indicating that if her card is j, she calls with probability q_j, and folds with probability $1 - q_j$. Similarly, a strategy for Player III is represented by a vector $\mathcal{R} = [r_1, \ldots, r_n]$, following the same interpretation as Player II.

We first define two procedures:

- `Call2` is used to calculate the payoff if either Player II or Player III decides to fold, leaving only two players (one of whom is Player I) to compare their cards. Let us assume that Player III folds. Then,

$$\texttt{Call2}(i, j, R) = \begin{cases} R+1 & \text{if } i > j \\ -R & \text{if } i < j. \end{cases}$$

- `Call3` is used to calculate the payoff when all the three players are comparing their cards:

$$\texttt{Call3}(i, j, k, R) = \begin{cases} 2R & \text{if } i > j \text{ and } i > k \\ -R & \text{if } i < j \text{ or } i < k. \end{cases}$$

The Fast NLP contains three sets of constraints, one set for each player, corresponding to the expected payoff over the pairs of distributions $\mathcal{Q} - \mathcal{R}$, $\mathcal{P} - \mathcal{R}$, or $\mathcal{P} - \mathcal{Q}$. For each player $l = 1, 2, 3$, there are two sets of constraints depending on the card that Player l has and whether they follow their first strategy or the second strategy:

$$\text{Minimize } \frac{1}{n}\sum_{c=1}^{n} v_c^1 + \frac{1}{n}\sum_{c=1}^{n} v_c^2 + \frac{1}{n}\sum_{c=1}^{n} v_c^3$$

subject to

$$\frac{1}{(n-1)(n-2)} \sum_{j \neq i} \sum_{k \neq i,j} \texttt{Call3}(i, j, k, 1) \leq v_i^1 \quad i = 1, \ldots, n \qquad \text{(Player I checks)}$$

$$\frac{1}{(n-1)(n-2)} \Bigg(\sum_{j \neq i} \sum_{k \neq i,j} \texttt{Call3}(i, j, k, b+1) \cdot q_k \cdot r_k$$
$$+ \texttt{Call2}(i, j, b+1) \cdot q_j \cdot (1 - r_k) + \texttt{Call2}(i, k, b+1) \cdot (1 - q_j) \cdot r_k$$
$$+ 2(1 - q_j) \cdot (1 - r_k) \Bigg) \leq v_i^1 \quad i = 1, \ldots, n \qquad \text{(Player I bets)}$$

$$\frac{1}{(n-1)(n-2)} \sum_{i \neq j} \sum_{k \neq i,j} (-p_i + \texttt{Call3}(j, i, k, 1) \cdot (1 - p_i)) \leq v_j^2 \quad j = 1, \ldots, n$$
$$\text{(Player II folds)}$$

$$\frac{1}{(n-1)(n-2)} \Bigg(\sum_{i \neq j} \sum_{k \neq i,j} \texttt{Call3}(j, i, k, b+1) \cdot p_i \cdot r_k$$

$$+ \texttt{Call2}(j,i,b+1)\cdot p_i\cdot(1-r_k)+\texttt{Call3}(j,i,k,1)\cdot(1-p_i)\Bigg) \le v_j^2 \quad j=1,\ldots,n$$

(Player II calls)

$$\frac{1}{(n-1)(n-2)}\sum_{i\neq k}\sum_{j\neq i,k}\left(-p_i+\texttt{Call3}(k,i,j,1)\cdot(1-p_i)\right)\le v_k^3 \quad k=1,\ldots,n$$

(Player III folds)

$$\frac{1}{(n-1)(n-2)}\Bigg(\sum_{i\neq k}\sum_{j\neq i,k}\texttt{Call3}(k,i,j,b+1)\cdot p_i\cdot q_j$$

$$+ \texttt{Call2}(k,i,b+1)\cdot p_i\cdot(1-q_j)+\texttt{Call3}(k,i,j,1)\cdot(1-p_i)\Bigg) \le v_k^3 \quad k=1,\ldots,n$$

(Player III calls)

$$0\le p_i,q_j,r_k\le 1 \quad i,j,k=1,\ldots,n.$$

Here, we assume that Players II and III adopt identical strategies. For example, with bet size 1 and $n=4$, we obtain the following results.

- Player I's strategy is: If your card is 1, bet with probability of $\frac{2}{3}$ and check with probability $\frac{1}{3}$. If your card is 2 or 3, then definitely checks; if your card is 4, definitely bet.
- Player II's and III's strategies are: If their card is 1 or 2, they definitely fold. If their card is 3, they call with probability of $\frac{1}{4}$ and fold with probability $\frac{3}{4}$. If their card is 4, they definitely call.
- The value of the game (for Player 1) is $\frac{1}{24}$, while for Players II and III are $-\frac{1}{48}$ each.

As a final remark, our model is a direct extension of von Neumann's two-player finite poker payoff matrix to three players, where Players II and III react to Player I, sharing identical strategies and payoffs. Therefore, it does not quite reflect a real 3-player poker game, where Player III gains an advantage from sequential play. However, sequential play could be incorporated by expanding Player III's strategies to account for Player II's potential call or fold. While this would complicate the payoff matrix, it is feasible.

Extension of von Neumann's continuous model to three players

As you may recall, we introduced this work with von Neumann's concept of an uncountably infinite deck, contrasting it with the finite nature of real-world card games. We proceeded by solving the finite deck games for two players and extended our analysis to include three players. Now that we have solved the finite version, the solutions effortlessly transition us to the continuous version of the three-player game, as we will demonstrate—a fitting conclusion to our study.

As in the von Neumann model for two players, each of the three players contributes 1 dollar to the pot and receives independent uniform(0,1) hands. As a reminder, Player I has the option to check or bet a fixed amount b, while Players II and III can only call or fold. The betting tree remains the same as that of the finite deck model. The conjectured Nash equilibrium strategies for three players, *guided by the data generated from the finite deck model*, are illustrated in the right panel of Figure 5. Please scroll up a few pages.

Our advice

For numbers A, B, C, yet *to be determined*,

- Player I: If $0 < x < A$ or $B < x < 1$ he should **bet**, otherwise **check**.
- Players II and III: If $0 < y < C$ they should **fold**, otherwise **call**.

To solve for the Nash equilibrium strategies, we apply the *Principle of Indifference*, which states that in mixed strategy Nash equilibria, players are indifferent between pure strategies as they yield the same expected payoff. This principle guides the solution for the three-player game: Players II and III adjust their strategies (y and z) to make Player I indifferent between his pure strategies. Similarly, Players I and III adjust their strategies to make Player II indifferent between her pure strategies, and so on.

Assume $0 < A < C < B$. We now determine the cut points A, B, and C by solving three indifference equations as follows.

1. **For Player I to be indifferent at A:**
   (a) If Player I checks at $x = A$, his expected payoff is

$$\int_0^A \int_0^A 2dzdy + \int_0^A \int_A^1 -1dzdy + \int_A^1 \int_0^1 -1dzdy$$

   (b) If Player I bets at $x = A$, his expected payoff is

$$\int_0^C \int_0^C 2dzdy + \int_0^C \int_C^1 -(b+1)dzdy + \int_C^1 \int_0^1 -(b+1)dzdy$$

   Equating the two expressions above yields the following equation:

$$3A^2 - 1 = 3C^2 + bC^2 - b - 1. \qquad \text{(Eq. A)}$$

2. **For Player I to be indifferent at B:**
   (a) If Player I checks at $x = B$, his expected payoff is

$$\int_0^B \int_0^B 2dzdy + \int_0^B \int_B^1 -1dzdy + \int_B^1 \int_0^1 -1dzdy$$

(b) If Player I bets at $x = B$, his expected payoff is

$$\int_0^C \int_0^C 2dzdy + \int_0^C \int_C^B (b+2)dzdy + \int_C^B \int_0^C (b+2)dzdy + \int_C^B \int_C^B 2(b+1)dzdy$$
$$+ \int_B^1 \int_0^B -(b+1)dzdy + \int_0^B \int_B^1 -(b+1)dzdy + \int_B^1 \int_B^1 -(b+1)dzdy$$

Equating the two expressions above yields the following equation:

$$3B^2 - 1 = -2bCB + 3bB^2 + 3B^2 - b - 1. \qquad \text{(Eq. B)}$$

3. **For Player II (or Player III) to be indifferent at C:**
   Assuming Player I bets:
   (a) If Player II folds at $y = C$, her expected payoff is

$$\int_0^A \int_0^1 -1dzdx + \int_B^1 \int_0^1 -1dzdx$$

   (b) If Player II calls at $y = C$, her expected payoff is

$$\int_0^A \int_0^C (b+2)dzdx + \int_0^A \int_C^1 -(b+1)dzdx + \int_B^1 \int_0^1 -(b+1)dzdx$$

   Equating the two expressions above yields the following equation:

$$-A + B - 1 = 2bCA + 3CA - bA - A - b + bB + B - 1. \qquad \text{(Eq. C)}$$

Solving the above **non-linear** system of three equations in three unknowns gives us the solutions for A, B and C for the Nash equilibrium strategies.

In particular, when $b = 2$, the optimal cuts are

$$A = 0.137058194328370$$
$$B = 0.829422249795391$$
$$C = 0.641304115985175.$$

This results in the value of the game (for Player I) being 0.122557074714865.

We can also determine the best bet amount b, that maximizes Player I's payoff under the Nash equilibrium strategies. Approximately, $b^* \approx 2.07$, resulting in Player I achieving a maximum payoff of 0.122590664136184. Therefore, we observe that the highest payoff for Player I in the three-player game exceeds that of the von Neumann's two-player game, which is $1/9 = 0.111111$ achieved at $b^* = 2$.

One final remark is that the Nash equilibrium for the three-player continuous game resembles those observed in the discrete model when n is large. In our experiments with the finite deck model, we we are able to simulate up to $n = 65$. With $b = 2$, we obtain the following results:

- The cuts are: $A = (8 + 14/23)/65 = 0.132441471571906$
  $B = 1 - 11/65 = 0.830769230769231$
  $C = 1 - (22 + 189/205)/65 = 0.647354596622889.$
- The value of the game (for Player I) is $\frac{974}{8121} = 0.119935968476789.$

The authors stand in front of John von Neumann's former residence at 26 Westcott Road, Princeton. The photo, taken on May 31, 2024, was courtesy of Karen Reid, the current owner.

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