Von Neumann and Newman Pokers with Finite Decks

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Welcome to the world of poker, where strategy and probability rule. Picture yourself at the poker table, every decision a crucial step toward victory or defeat. Poker has intrigued mathematicians for decades as a window into decision-making and game theory. Pioneers like Émile Borel, John von Neumann, Harold W. Kuhn, John Nash, and Lloyd Shapley [1, 4, 6, 8], who believed that real-life scenarios mirror poker with their elements of bluffing and strategic thinking, have simplified the complexities of the game, making it tractable for game-theoretic analysis.

A Quick Game Theory Refresher

In game theory, a game refers to a situation in which players make decisions that result in outcomes based on the choices of all involved. A strategy is a complete plan of action that a player will follow in various situations throughout the game. A pure strategy is a strategy in which a player makes a specific choice or takes a specific action with certainty whenever a particular situation arises in the game. When players use a mixed strategy, instead of choosing a single, deterministic action, they randomize over possible moves, assigning a probability to each move. A Nash equilibrium occurs when no player can benefit from changing their strategy while the other players keep theirs unchanged. A mixed Nash equilibrium is a type of Nash equilibrium such that at least one player uses a mixed strategy, ensuring that no player can improve their payoff by individually changing their strategy.

Von Neumann Poker

In its original version [8], von Neumann (see Figure 1) proposed and solved the following game of poker with an uncountably infinite deck containing all the real numbers between 0 and 1. Fix a bet size b. Player I and Player II are dealt (uniformly at random) two "cards," real numbers x and y, in the interval $[0, 1]$. They each see their own card but have no clue about the opponent's card. At the start, they each put one dollar (the ante) into the pot. So now the pot has two dollars.

Figure 2 illustrates the "betting tree" of this game. Here, Player I looks at his card and decides whether to call "check," in which case each player shows their card and whoever has the larger card wins the pot. On the other hand, he has the option to place a bet, putting b additional dollars in the pot. Now the game turns to Player II. She can decide to fold, in which case player I gets the pot, gaining one dollar (with a loss of one dollar for player II), or she could be brave and call, putting her own b dollars into the pot, which now has $2b + 2$ dollars. The cards are compared in a showdown, and whoever has the higher card wins the whole pot, resulting in a gain of $b + 1$ dollars for the winner and a loss of $b + 1$ for the loser.

John von Neumann proved that the following pair of strategies is a pure Nash equilibrium, i.e., if each player follows their chosen strategy, neither of them can do better (on average) by adopting a different strategy.

The von Neumann Advice

Von Neumann identified the cuts A, B, C in the right panel of Figure 2 and proposed the following strategies:

- Player I: If

$$0 < x < \frac{b}{(b+4)(b+1)} \quad \text{or} \quad \frac{b^2+4b+2}{(b+4)(b+1)} < x < 1,$$

bet; otherwise, check.
- Player II: If

$$0 < y < \frac{b(b+3)}{(b+4)(b+1)},$$

fold; otherwise, call.

Note that Player II's strategy corresponds to common sense: there is some cutoff below which one should be conservative and cut your losses, giving up the one dollar and not risking losing b additional dollars, and above which one should be brave and go for it.

Now, common sense might also tell you that Player I should also have his own cutoff, checking if his card is below it and betting if it exceeds it. But this is not the optimal strategy. If Player I has a low card, he should bluff, acting as though he had a high card, thereby intimidating player II into folding. Sad but true, honesty is not always the best policy. Indeed, the game favors Player I, whose expected gain is

$$\frac{b}{(b+4)(b+1)}.$$

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Figure 1. The authors stand in front of John von Neumann's former residence at 26 Westcott Road, Princeton, New Jersey. The photo, taken on May 31, 2024, is courtesy of Karen Reid, the current owner.

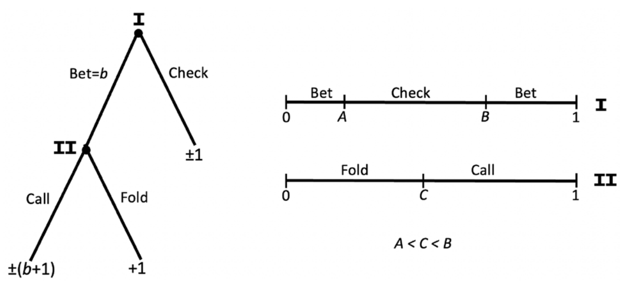

Figure 2. The betting tree and Nash equilibrium strategies for von Neumann poker.

Strategy	$S_2 = \{\ \}$ Always Fold	$S_2 = \{1\}$ Call if "1", Fold if "2"	$S_2 = \{2\}$ Call if "2", Fold if "1"	$S_2 = \{1{,}2\}$ Always Call	Row Min
$S_1 = \{\ \}$ Always Check	(-1+1)/2 = 0	(-1+1)/2 = 0	(-1+1)/2 = 0	(-1+1)/2 = 0	0
$S_1 = \{1\}$ Bet if "1", Check if "2"	(+1+1)/2 = 1	(+1+1)/2 = 1	(-3+1)/2 = -1	(-3+1)/2 = -1	-1
$S_1 = \{2\}$ Bet if "2", Check if "1"	(-1+1)/2 = 0	(-1+3)/2 = 1	(-1+1)/2 = 0	(-1+3)/2 = 1	0
$S_1 = \{1{,}2\}$ Always Bet	1	(+1+3)/2 = 2	(-3+1)/2 = -1	(-3+3)/2 = 0	-1
Column Max	1	2	0	1	

Figure 3. Payoff matrix for $n = 2$ and $b = 2$ along with the values of the row minima and column maxima.

When the bet size b equals 2, the advice spells out as follows:

- Player I: if $0 < x < 1/9$ or $7/9 < x < 1$, bet; otherwise, check.
- Player II: If $0 < y < 5/9$, fold; otherwise, call.

The expected value for Player I, i.e., the value of the game, is 1/9. It can be shown that $b = 2$ maximizes Player I's payoff under the Nash equilibrium strategies.

Finitely Many Cards

What we don't like about von Neumann's original version is that the deck is infinite. In real life, decks have only finitely many cards, and in fact, not that many. We were wondering whether there exist pure Nash equilibria when there are only finitely many cards.[1]

Finding all pure Nash equilibria via von Neumann's minimax theorem. Let $n \geq 2$ be a fixed positive integer representing the number of cards in the deck, numbered $1, 2, \ldots, n$. Additionally, let $b \geq 1$ be a fixed positive integer denoting the bet size. In this section, we aim to identify the set of all pure Nash equilibria for a given pair (n, b), which, as we will see, may occasionally be empty.

Since we did not make any a priori assumptions about plausible strategies, a strategy for Player I can be any subset S_1 of $\{1, \ldots, n\}$ that states, "If your card belongs to S_1, you should bet; otherwise, check." Similarly, a strategy S_2 for Player II can be any such subset that tells her to call if and only if her card j is in S_2. We then construct a pay table, which is a $2^n \times 2^n$ payoff matrix. The smallest payoff matrix of size 4×4 for a bet size $b = 2$ and $n = 2$ cards is given in Figure 3.

To solve for a pure Nash equilibrium in our finite poker game, it is fitting to revisit John von Neumann's minimax theorem, first published in 1928 [7]. This theorem remains a cornerstone of game theory to this day, and it is a privilege to apply von Neumann's celebrated result to solve the finite version of his poker model.

In the context of a two-player zero-sum game, given a payoff matrix, the theorem states that if the row maximin equals the column minimax, then a pure Nash equilibrium (or saddle point) is guaranteed to exist. In particular, at an equilibrium strategy pair $[S_1, S_2]$, Player I (the row player) aims to maximize his worst possible payoff, while Player II (the column player) seeks to minimize Player I's best payoff. The value of the game (the expected gain of Player I and the corresponding loss of Player II) is the outcome at this equilibrium pair, representing the optimal result for both players.

However, if these values do not coincide, the equilibrium will typically involve mixed strategies, whereby both players randomize their decisions. Here is a historical note related to this: von Neumann and Oskar Morgenstern showed that for a zero-sum game, there must always exist at least one mixed Nash equilibrium, as demonstrated in their 1944 book *Theory of Games and Economic Behavior* [8].

[1] Before diving into the world of finite poker, the interested reader can find detailed implementations of this work in the Maple package available at https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/poker.html, which also includes links to numerous output files, as well as an expanded version of the current work from a computational perspective in [3].

In 1950, John Nash generalized this concept in his paper "Non-Cooperative Games" [5] to the non-zero-sum game.

We will talk about mixed Nash equilibria later on. But for now, let's explore the pure ones. Referring to Figure 3, we see that for the payoff matrix when the bet size is $b = 2$ and we have only two cards, the value of the row maximin equals the column minimax, both being 0, at two pairs $[S_1, S_2]$ of pure Nash equilibria:

$$[\{\}, \{2\}] \quad \text{and} \quad [\{2\}, \{2\}].$$

In both of them, Player II calls if her card is 2 and folds if her card is 1, while Player I always checks in the first strategy, and checks if his card is 1 in the second strategy. This is not very interesting, since the value of the game is 0.

Fixing the bet size at 2 and setting $n = 3$ cards (so the payoff matrix is now 8×8) is, frankly, a bit dull, since it leads to just two trivial pairs of pure Nash equilibria: $[\phi, \{3\}]$ and $[\{3\}, \{3\}]$. Increasing the number of cards to four, five, or even six (while the size of the payoff matrix grows exponentially) doesn't add much excitement—they are all empty, since the value of the row maximin does not equal the column minimax, resulting in no pure Nash equilibria.

But now comes a nice surprise. With seven cards, we get not one, not two, but three pure, nontrivial Nash equilibria! In all of them, Player I bets if his card is in $\{1, 6, 7\}$, while Player II calls if her card belongs to any of the following sets: $\{3, 6, 7\}$, $\{4, 6, 7\}$, $\{5, 6, 7\}$. The value of the game is 2/21. So with seven cards, we already have bluffing! If Player I has the card labeled 1, he should bet even though he will definitely lose the bet if Player II calls.

Moving right along, with eight cards, we also get three pure Nash equilibria. For all of them, Player I bets iff his card belongs to $\{1, 7, 8\}$, but Player II calls if her card is in one of $\{4, 7, 8\}$, $\{5, 7, 8\}$, $\{6, 7, 8\}$. The value of the game is 3/28, getting tantalizingly close to von Neumann's 1/9.

Since the sizes of the payoff matrices grow exponentially and we did not make any plausibility assumptions, we can go only so far with this naive "vanilla" approach. But nine cards are still doable. Indeed, there are seven pure Nash equilibria in this case. For all of them, we have $S_1 = \{1, 8, 9\}$, but Player II has seven choices, all with four members, including, of course, $\{6, 7, 8, 9\}$.

To overcome the exponential explosion, we can stipulate that Player I's strategy must be of the form "Check iff $i \in \{A, A+1, \ldots, B\}$ for some $1 \le A < B \le n$," while Player II's must be of the form "Call iff $j \in \{C, C+1, \ldots, n\}$ for some $1 \le C \le n$."

Now we can go much further, which leads to a nice result: If n is a multiple of 9, then the (restricted) pure Nash equilibria are as expected. Namely, the value of the game is 1/9 and the strategy for player I is to check if

$$\frac{1}{9}n < i \le \frac{7}{9}n$$

and bet otherwise, while for Player II it is call iff

$$j > \frac{5}{9}n.$$

If n is not a multiple of 9, then the values are close, but a little less so. For example, for $n = 26$, the value is $36/325 = 0.110769$. For $n = 25$, the value is $11/100 = 0.11$.

Mixed Nash Equilibria via Linear Programming

The study of mixed strategies in two-person zero-sum games can be elegantly formulated as a primal–dual linear programming (LP) problem. A mixed strategy involves each player choosing optimal actions according to a probability distribution, introducing uncertainty. An equilibrium solution to this dual pair of linear programs reveals optimal mixed strategies (mixed Nash equilibria) for both players.

Slow linear programming for mixed Nash equilibria. Recall our scenario: the pot starts at $1 + 1$, with only Player I able to bet a fixed amount b. Given the $2^n \times 2^n$ payoff matrix (m_{ij}) as input, Player I aims to maximize his worst-case expected gain, minimizing over all possible actions of Player II. This objective is framed as a linear programming problem by introducing the variable v_1 to represent this minimum, ensuring that Player I's expected gain is at least v_1 for every action of Player II, and maximizing v_1. Similarly, from Player II's viewpoint, the goal is to minimize her worst-case expected loss, maximizing over all actions of Player I. This involves introducing the variable v_2 to represent this maximum and setting the objective to minimize v_2.

To formulate the primal–dual linear programming problem, let $\mathbf{x} = (x_1, \ldots, x_{2^n})$ be the mixed strategy probability of Player I to maximize v_1. Let $\mathbf{y} = (y_1, \ldots, y_{2^n})$ be the mixed strategy probability of Player II to minimize v_2.

Primal: Maximize v_1 such that

$$\sum_{i=1}^{2^n} x_i \cdot m_{ij} \ge v_1 \quad \text{for } j = 1, \ldots, 2^n,$$
$$\sum_{i=1}^{2^n} x_i = 1, \ \text{ s.t. } x_i \ge 0 \text{ for } i = 1, \ldots, 2^n.$$

Dual: Minimize v_2 such that

$$\sum_{j=1}^{2^n} m_{ij} \cdot y_j \le v_2 \quad \text{for } i = 1, \ldots, 2^n,$$
$$\sum_{j=1}^{2^n} y_j = 1, \ \text{ s.t. } y_j \ge 0 \quad \text{for } j = 1, \ldots, 2^n.$$

By the minimax theorem, at an equilibrium, we have $v_1 = v_2 = v^*$, which represents the value of the game. For example, with $n = 4$ cards and a bet size $b = 2$, one can set up a 16×16 payoff matrix and solve the above linear programming problems for a mixed-strategy Nash equilibrium, resulting in the following:

- Player I has two strategies: (1.1) with probability 1/2, bet if his card is 4 and fold if his card is 1, 2, or 3; and (1.2) with probability 1/2, bet if his card is 1 or 4, and fold if his card is 2 or 3.
- Player II has two strategies: (2.1) with probability 1/2, call if her card is 4 and fold if her card is 1, 2, or 3; and (2.2) with probability 1/2, call if her card is 2 or 4, and fold if her card is 1 or 3.
- The value of the game is 1/12.

However, due to the exponentially large size of the matrix, practical limitations arise that prevent us from considering more than six or seven cards without the inconvenience of reducing the dominated rows and columns of the payoff matrix.

Fast linear programming problem for mixed Nash equilibria. The Nash equilibrium can be considered from a different perspective, where focusing on the specific card each player receives reduces the number of constraints from exponential to linear. A strategy for Player I is given by a vector $\mathcal{P} = [p_1, \ldots, p_n]$ that tells him that if his card is i, bet with probability p_i, and check with probability $1 - p_i$. A strategy for Player II is given by a vector $\mathcal{Q} = [q_1, \ldots, q_n]$ that tells her that if her card is j, call with probability q_j and fold with probability $1 - q_j$.

Before we discuss the fast linear programming formulation, let us mention that given card-by-card strategies $\mathcal{P}$ and $\mathcal{Q}$, it is easy to compute the expected payoff (for Player I) as a bilinear form in the p_i and q_j:

$$\begin{aligned}&\texttt{Payoff}(n, b, \mathcal{P}, \mathcal{Q})\\ &= \frac{1}{n(n-1)}\Bigg(\sum_{i=1}^{n}\sum_{j=1}^{i-1}(1-p_i) - \sum_{i=1}^{n}\sum_{j=i+1}^{n}(1-p_i)\\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{i-1}p_i(1-q_j) + \sum_{i=1}^{n}\sum_{j=i+1}^{n}p_i(1-q_j)\\ &\quad + (b+1)\sum_{i=1}^{n}\sum_{j=1}^{i-1}p_iq_j - (b+1)\sum_{i=1}^{n}\sum_{j=i+1}^{n}p_iq_j\Bigg).\end{aligned}$$

Let us now return to the fast linear programming problem for Player I, which contains two sets of constraints. Each set corresponds to the expected payoff (over distribution $\mathcal{P}$), conditioned on the card that Player II has and whether she calls or folds:

$$\begin{aligned}&\text{Maximize } \frac{1}{n}\sum_{j=1}^{n} v_j \text{ s.t.}\\ &\frac{1}{n-1}\sum_{i\neq j}\left(\texttt{Call}(i,j,b+1)\cdot p_i + \texttt{Call}(i,j,1)\cdot(1-p_i)\right) \geq v_j,\\ &\quad j = 1, \ldots, n, \text{ (Player II calls)}\\ &\frac{1}{n-1}\sum_{i\neq j}\left(p_i + \texttt{Call}(i,j,1)\cdot(1-p_i)\right) \geq v_j\\ &\quad j = 1, \ldots, n, \text{ (Player II folds)}\\ &\quad 0 \leq p_i \leq 1, \quad i = 1, \ldots, n,\end{aligned} \tag{VN-I}$$

where the procedure $\texttt{Call}(i, j, R)$ is defined based on whether the card i is larger than card j:

$$\texttt{Call}(i,j,R) = \begin{cases} R & \text{if } i > j,\\ -R & \text{if } i < j.\end{cases}$$

Similarly, for the fast linear programming problem for Player II, the constraints are calculated based on the expected loss (over the distribution $\mathcal{Q}$), conditioned on the card that Player I has and whether he bets or checks:

$$\begin{aligned}&\text{Minimize } \frac{1}{n}\sum_{i=1}^{n} v_i \text{ s.t.}\\ &\frac{1}{n-1}\sum_{j\neq i}\left(\texttt{Call}(i,j,b+1)\cdot q_j + (1-q_j)\right) \leq v_i,\\ &\quad i = 1, \ldots, n, \text{ (Player I bets)}\\ &\frac{1}{n-1}\sum_{j\neq i}\texttt{Call}(i,j,1) \leq v_i \quad i = 1, \ldots, n, \text{ (Player I checks)}\\ &\quad 0 \leq q_j \leq 1, \quad j = 1, \ldots, n.\end{aligned} \tag{VN-II}$$

Now things get interesting much sooner. Even with just three cards, we already have bluffing! With a bet size of 1 (note the difference in bet size from the usual $b = 2$), the results are as follows:

- Player I's strategy is if his card is 1, bet with probability 1/3 and check with probability 2/3. If his card is 2, then definitely check, while if it is 3, then he should definitely bet.
- Player II's strategy is if her card is 1, definitely fold, while if it is 2, call with probability 1/3 and fold with probability 2/3, while if the card is 3, then definitely call.
- The value of the game is $1/18 = 0.055555\ldots$.

So already with three cards, Player I should sometimes bluff if his card is 1, but only with probability 1/3.

Note that a pure Nash equilibrium is also a mixed one, and indeed, in some cases, we obtain pure Nash equilibria. For example, with $n = 18$ cards and $b = 2$, the result is as follows:

- Player I: Bet iff your card is in $\{1, 2, 15, 16, 17, 18\}$.
- Player II: Call iff your card is in $\{11, \ldots, 18\}$.
- The value of the game is $\frac{1}{9} \approx 0.111111111\ldots$.

Beyond the results obtained for the more realistic scenario of finitely many cards and their computational efficiency of the fast linear programming model, our findings provide crucial insights into the continuous case, shedding light on why Player I's Nash equilibrium strategy in von Neumann's poker follows the pattern depicted in Figure 2: bluffing when his card is small.

D. J. Newman Poker

Not as famous as John von Neumann, but at least as brilliant, is Donald J. Newman, the third person to be a Putnam fellow in three consecutive years. He was a good friend of

John Nash. In a fascinating four-page paper [9] in *Operations Research*, he proposed his own version of poker in which the bet size is not fixed but can be decided by Player I, including betting 0, which is the same as checking.

In his own words (now the players are A and B):

> A and B each ante 1 dollar and are each dealt a "hand," namely a randomly chosen real number in (0, 1). Each sees his, but not the opponent's, hand. A bets any amount he chooses (≥ 0); B "sees" him (i.e., calls, betting the same amount) or folds. The payoff is as usual.

The D. J. Newman Advice

The betting tree and Nash equilibrium strategies are the same as those in von Neumann Poker, as shown in Figure 2. However, Player I is allowed to bet with different positive amounts.

Player I's strategy is as follows (see also Figure 4).

- Case 1: For a given card $x < A = 1/7$, Player I should bet an amount R that satisfies the relation

$$x = \frac{1}{7} - \frac{R^2(R+6)}{7(R+2)^3} = \frac{4(3R+2)}{7(R+2)^3}.$$

- Case 2: For a given card $1/7 = A < x < B = 4/7$, Player I should check.
- Case 3: For a given card $x > B = 4/7$, Player I should bet an amount R that satisfies the relation

$$x = \frac{R^2 + 4R + 2(1/7) + 2}{(R+2)^2} = \frac{7R^2 + 28R + 16}{7(R+2)^2}.$$

And here is Player II's strategy (see also Figure 5). In response to a bet amount $R > 0$ from Player I, Player II should fold if her card satisifes $y < C$ and call if $y > C$, where

$$C = \frac{R + 2(\frac{1}{7})}{R+2} = \frac{7R+2}{7R+14}.$$

The value of the game (for Player I) is $1/7$. (This is greater than $1/9$ in von Neumann's game, since Player I has more freedom to bet.) The detailed calculation of this result can be found in Newman's original paper [9], or check our supplementary material at https://thotsaporn.com/SupplementN.pdf.

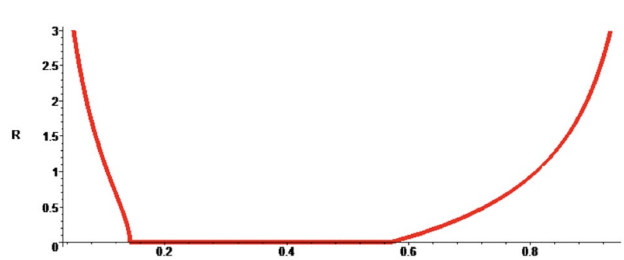

Figure 4. Relation between the value of the card x and the optimal bet size R for Player I.

Finitely Many Cards

But in real life, there is always a finite number of cards, and no one can bet arbitrarily large amounts. Once again, we focus on the finite-deck version, which is set up as follows: The inputs are integers $n \geq 2$ and $b \geq 1$, where each player is dealt a different card from $\{1, \ldots, n\}$, and Player I's decision, on seeing his card i, is to choose an amount s to bet from $\{0, \ldots, b\}$, where $s = 0$ corresponds to checking.

In this game, the number of strategies is even larger, and we will not bother with the vanilla approach to find pure Nash equilibria. Instead, we will look for (fast linear programming) mixed strategies right away.

Player I's payoff maximization. Player I's strategy space consists of an $n \times (b+1)$ matrix $\left(p_i[s]\right)$, where $p_i[s]$ $(1 \leq i \leq n, 0 \leq s \leq b)$ is the probability that if he has card i, he will bet s dollars (of course, the row sums should add up to 1). The linear programming formulation is analogous to that of (VN-I) in the previous section. Let us point out the differences to gain some insights. Recall that each constraint corresponds to the card that Player II has and her choice of action. In (VN-I), Player II can either call or fold, and she can have one of the n cards. Hence, there is a total of $2n$ constraints.

In our current scenario, however, Player II's decision depends on both her card and Player I's proposed bet amount s. Let $S_b := \{0, \ldots, b\}$. We define $\mathcal{P}(S_b)$ as the set containing all possible strategies for Player II regarding whether to call or fold. That is, each $Y \in \mathcal{P}(S_b)$ represents a strategy whereby Player II will call if $s \in Y$. Therefore, for a fixed card j and strategy $Y \in \mathcal{P}(S_b)$ of Player II, the constraint is that Player II will call if she holds card j and the proposed bet amount s is in Y; otherwise, she folds. The total number of these constraints is $n \cdot 2^b$.

For example, when $b = 4$, we have

$$\begin{aligned}\mathcal{P}(S_4) = \{&\{0\}, \{0,1\}, \{0,2\}, \{0,3\}, \{0,4\}, \{0,1,2\}, \{0,1,3\},\\ &\{0,1,4\}, \{0,2,3\}, \{0,2,4\}, \{0,2,3,4\}, \{0,1,2,3,4\}\\ &\{0,3,4\}, \{0,1,2,3\}, \{0,1,2,4\}, \{0,1,3,4\}\}.\end{aligned}$$

With this setup, we derive the following linear programming problem:

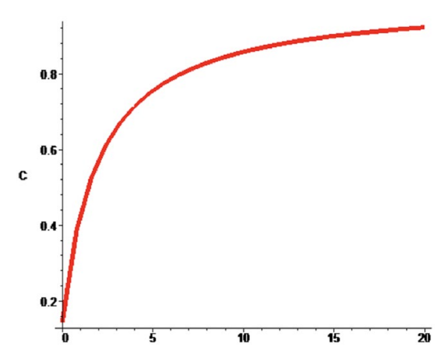

Figure 5. Relation between the amount of bet R and the value of the card C that is sufficient to call the bet for Player II.

$$\text{Maximize } \frac{1}{n}\sum_{j=1}^{n} v_j \text{ s.t.}$$

$$\frac{1}{n-1}\sum_{i\neq j}\left(\sum_{s\in Y}\texttt{Call}(i,j,s+1)\cdot p_i[s] + \sum_{s\in(S_b\setminus Y)} p_i[s]\right) \geq v_j,$$

$$\underbrace{j=1,\dots,n;\; Y\in\mathcal{P}(S_b),}_{\text{total } n\cdot 2^b \text{ constraints}}$$

$$\sum_{s=0}^{b} p_i[s] = 1, \quad i=1,\dots,n,$$

$$p_i[s]\geq 0, \quad s=0,\dots,b;\, i=1,\dots,n. \qquad \text{(DJN-I)}$$

Player II's loss minimization. While Player's II's strategy is also an $n\times(b+1)$ matrix, formulating the linear programming problem is much simpler. Let's denote the matrix by $(q_j[s])$, where $q_j[s]$ is the probability of Player II's calling if her card is j and the bet proposed by Player I is s (and as usual, $1-q_j[s]$ is the corresponding probability of folding). In this case, there is a total of $n(b+1)$ constraints (not exponential, as in the case of Player I). Also, the LP formulation straightforwardly extends from (VN-II):

$$\text{Minimize } \frac{1}{n}\sum_{i=1}^{n} v_i \text{ s.t.}$$

$$\frac{1}{n-1}\sum_{j\neq i}\left(\texttt{Call}(i,j,s+1)\cdot q_j[s] + (1-q_j[s])\right) \leq v_i,$$

$$\underbrace{s=0,\dots,b;\, i=1,\dots,n,}_{\text{total } n(b+1) \text{ constraints}}$$

$$q_j[0]=1, \quad j=1,\dots,n,$$

$$0\leq q_j[s]\leq 1, \quad s=0,\dots,b;\, j=1,\dots,n. \qquad \text{(DJN-II)}$$

For example, for the game with $n=7$ cards and the maximum bet size $b=3$, the value of this game is $13/105=0.1238095238$, which is still less than $1/7=0.1428571429$ (due to the limitation of the maximum bet amount). Optimal strategies for both players are given by the following matrices:

$$(p_i^*[s])_{7\times 4} = \begin{bmatrix} \frac{1}{15} & \frac{1}{3} & 0 & \frac{3}{5} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \left(q_j^*[s]\right)_{7\times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & \frac{1}{10} & 0 \\ 1 & 1 & 1 & \frac{2}{5} \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

We will now interpret the optimal strategies of Players I and II through the obtained $p_i^*[s]$ and $q_j^*[s]$. For Player I, if he holds card 1, his optimal bet amounts will be 0, 1, 2, and 3, with corresponding probabilities of $1/15$, $1/3$, 0, $3/5$, respectively. If his card falls between 2 and 5, he will always check. If his card is 6, he will bet 1 with certainty, and if his card is 7, he will place the maximum bet of 3.

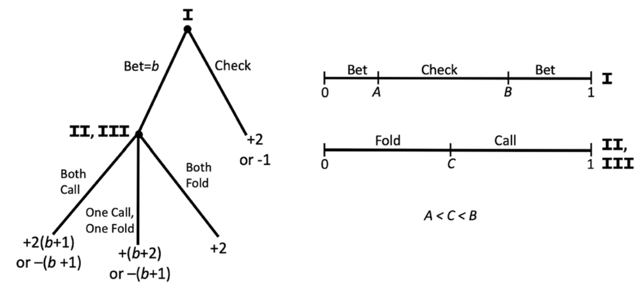

Figure 6. Three-player poker. Left: the betting tree. Right: conjectured Nash equilibrium strategies for a continuous deck.

On the other hand, to interpret Player II's strategy, we examine the matrix $q_j^*[s]$ column by column, corresponding to the proposed amounts bet by Player I. In response to Player I's actions, if Player I checks (i.e., proposes a bet amount of 0), Player II will always call, which corresponds to ones in the first column. If Player I bets 1 (as in column 2), Player II will fold if her card is less than 4 and call otherwise. A similar interpretation can be made for the remaining columns.

We noticed that for any given n, there exists a maximal bet size after which the game has the same value. As n grows larger and b reaches its saturation value, the value of the game seems to converge to the D. J. Newman "continuous" value $1/7$.

A Three-Player Poker Game

As early as 1950, future economics Nobel laureates John Nash and Lloyd Shapley [6] pioneered the analysis of a three-player poker game. They explored a simplified version in which the deck contains only two kinds of cards, High and Low, in equal numbers. Today, however, eighty years after von Neumann's analysis of poker, the dynamics of this three-player game remain unexplored. We now take the opportunity to analyze these dynamics in both their finite and infinite versions.

A Finite Deck

The three players each put one dollar into the pot. Player I acts first, choosing either to check or to bet a fixed integer amount $b>0$. If Player I checks, the three hands are immediately compared, and the player with the highest hand wins the pot. However, if Player I bets, Players II and III have two choices: call or fold. The reader is invited to refer to the left panel of Figure 6, which depicts the betting tree for three players. (The right panel shows the conjectured Nash equilibrium strategies to be used in the next section for the continuous version of the game.)

Assume that we are given three-dimensional payoff matrices $(M_l,\ l=1,2,3)$ for the three players:

$$M_l = \left(m_{ijk}^l\right),$$

where $i,j,k=1,2,\dots,2^n$.

While its counterpart two-player game can be solved using linear programming, here we require nonlinear programming (NLP) [2]. The NLP formulation for the three-player game closely follows the LP model for the two players discussed in the previous section. Each player aims to minimize their expected loss or the expected gain of the other players. For instance, given Player I's payoff matrix M_1, the other two players attempt to minimize the maximum potential loss incurred due to Player I's choices. This involves constraints that utilize matrix M_1 and the probability distributions $\mathbf{y} = \left(y_1, \ldots, y_{2^n}\right)$ and $\mathbf{z} = \left(z_1, \ldots, z_{2^n}\right)$ of Players II and III. These are embedded in the first set of constraints in the NLP formulation, which we will now formulate.

The slow NLP for three players is given by

$$\text{Minimize } \sum_{l=1}^{3} v^l \text{ s.t.}$$

$$\sum_{j,k=1}^{2^n} m^1_{ijk} \cdot y_j \cdot z_k \leq v^1 \quad \text{for } i = 1, 2, \ldots, 2^n,$$

$$\sum_{i,k=1}^{2^n} m^2_{ijk} \cdot x_i \cdot z_k \leq v^2 \quad \text{for } j = 1, 2, \ldots, 2^n,$$

$$\sum_{i,j=1}^{2^n} m^3_{ijk} \cdot x_i \cdot y_j \leq v^3 \quad \text{for } k = 1, 2, \ldots, 2^n,$$

$$\sum_{i=1}^{2^n} x_i = 1, \qquad \sum_{j=1}^{2^n} y_j = 1, \qquad \sum_{k=1}^{2^n} z_k = 1,$$

$$x_i, y_j, z_k \geq 0 \quad \text{for } i, j, k = 1, 2, \ldots, 2^n.$$

Note that if there are only two players, then z_k in the above NLP formulation disappears, and the constraint functions become linear in the variables x_i and y_j. Thus, the problem can be decomposed into two separate LP (primal–dual) problems, as discussed earlier.

We now shift our focus to the fast NLP formulation for three players, which aligns with the fast LP formulation for two players, considering the card that each player receives. Recall that a strategy for Player I is given by a vector $\mathcal{P} = [p_1, \ldots, p_n]$, indicating that if his card is i, he bets with probability p_i and checks with probability $1 - p_i$. A strategy for Player II is given by a vector $\mathcal{Q} = [q_1, \ldots, q_n]$, indicating that if her card is j, she calls with probability q_j and folds with probability $1 - q_j$. Similarly, a strategy for Player III is represented by a vector $\mathcal{R} = [r_1, \ldots, r_n]$, following the same interpretation as Player II's stragegy.

We first define two procedures:

- `Call2` is used to calculate the payoff if either Player II or Player III decides to fold, leaving only two players (one of whom is Player I) to compare their cards. Let us assume that Player III folds. Then

$$\texttt{Call2}(i, j, R) = \begin{cases} R+1 & \text{if } i > j, \\ -R & \text{if } i < j. \end{cases}$$

- `Call3` is used to calculate the payoff when all three players compare their cards:

$$\texttt{Call3}(i, j, k, R) = \begin{cases} 2R & \text{if } i > j \text{ and } i > k, \\ -R & \text{if } i < j \text{ or } i < k. \end{cases}$$

The fast NLP contains three sets of constraints, one set for each player, corresponding to the expected payoff over the pairs of distributions $\mathcal{Q}$—$\mathcal{R}$,$\mathcal{P}$—$\mathcal{R}$,$\mathcal{P}$—$\mathcal{Q}$. For each player $l = 1, 2, 3$, there are two sets of constraints depending on the card that Player l has and whether they follow their first strategy or the second strategy:

$$\text{Minimize } \frac{1}{n}\sum_{c=1}^{n} v^1_c + \frac{1}{n}\sum_{c=1}^{n} v^2_c + \frac{1}{n}\sum_{c=1}^{n} v^3_c$$

subject to

$$\frac{1}{(n-1)(n-2)} \sum_{j\neq i} \sum_{k\neq i,j} \texttt{Call3}(i, j, k, 1) \leq v^1_i, \quad i = 1, \ldots, n,$$

(Player I checks)

$$\frac{1}{(n-1)(n-2)} \Bigg(\sum_{j\neq i} \sum_{k\neq i,j} \texttt{Call3}(i, j, k, b+1) \cdot q_k \cdot r_k + \texttt{Call2}(i, j, b+1) \cdot q_j \cdot (1 - r_k) + \texttt{Call2}(i, k, b+1) \cdot (1 - q_j) \cdot r_k + 2(1 - q_j) \cdot (1 - r_k) \Bigg) \leq v^1_i, \quad i = 1, \ldots, n,$$

(Player I bets)

$$\frac{1}{(n-1)(n-2)} \sum_{i\neq j} \sum_{k\neq i,j} \left(-p_i + \texttt{Call3}(j, i, k, 1) \cdot (1 - p_i)\right) \leq v^2_j, \quad j = 1, \ldots, n,$$

(Player II folds)

$$\frac{1}{(n-1)(n-2)} \Bigg(\sum_{i\neq j} \sum_{k\neq i,j} \texttt{Call3}(j, i, k, b+1) \cdot p_i \cdot r_k + \texttt{Call2}(j, i, b+1) \cdot p_i \cdot (1 - r_k) + \texttt{Call3}(j, i, k, 1) \cdot (1 - p_i) \Bigg) \leq v^2_j, \quad j = 1, \ldots, n,$$

(Player II calls)

$$\frac{1}{(n-1)(n-2)} \sum_{i\neq k} \sum_{j\neq i,k} \left(-p_i + \texttt{Call3}(k, i, j, 1) \cdot (1 - p_i)\right) \leq v^3_k, \quad k = 1, \ldots, n,$$

(Player III folds)

$$\frac{1}{(n-1)(n-2)} \Bigg(\sum_{i\neq k} \sum_{j\neq i,k} \texttt{Call3}(k, i, j, b+1) \cdot p_i \cdot q_j + \texttt{Call2}(k, i, b+1) \cdot p_i \cdot (1 - q_j) + \texttt{Call3}(k, i, j, 1) \cdot (1 - p_i) \Bigg) \leq v^3_k, \quad k = 1, \ldots, n,$$

(Player III calls)

for $0 \le p_i, q_j, r_k \le 1, i, j, k = 1, \ldots, n$.

Here we assume that Players II and III adopt identical strategies. For example, with bet size 1 and $n = 4$, we obtain the following results.

- Player I's strategy is if his card is 1, bet with probability 2/3 and check with probability 1/3. If his card is 2 or 3, then definitely check; if his card is 4, definitely bet.
- The strategies of Players II and III are if their card is 1 or 2, they definitely fold. If their card is 3, they call with probability 1/4 and fold with probability 3/4. If their card is 4, they definitely call.
- The value of the game (for Player 1) is 1/24, while for each of Players II and III it is $-1/48$.

As a final remark, our model is a direct extension of von Neumann's two-player finite poker payoff matrix to three players, where Players II and III react to Player I, sharing identical strategies and payoffs. Therefore, it does not quite reflect a real three-player poker game, in which Player III would gain an advantage from sequential play. However, sequential play could be incorporated by expanding Player III's strategies to account for Player II's potential call or fold. While this would complicate the payoff matrix, it is feasible.

Extension of von Neumann's Continuous Model to Three Players

As you may recall, we introduced this article with von Neumann's concept of an uncountably infinite deck, contrasting it with the finite nature of real-world card games. We proceeded by solving the finite-deck games for two players and extended our analysis to include three players. Now that we have solved the finite version, the solutions effortlessly transition us to the continuous version of the three-player game, as we will demonstrate, giving us a fitting conclusion to our study.

As in the von Neumann model for two players, each of the three players contributes one dollar to the pot and receives independent uniform(0, 1) hands. As a reminder, Player I has the option to check or bet a fixed amount b, while Players II and III can only call or fold. The betting tree remains the same as that for the finite-deck model. The conjectured Nash equilibrium strategies for three players, guided by the data generated from the finite-deck model, are illustrated in the right panel of Figure 6.

Our Advice. For numbers A, B, C, yet to be determined,

- Player I: If $0 < x < A$ or $B < x < 1$, bet; otherwise, check.
- Players II and III: If $0 < y < C$, fold; otherwise, call.

To solve for the Nash equilibrium strategies, we apply the principle of indifference, which states that in mixed strategy Nash equilibria, players are indifferent between pure strategies, since they yield the same expected payoff. This principle guides the solution of the three-player game: Players II and III adjust their strategies (y and z) to make Player I indifferent between his pure strategies. Similarly, Players I and III adjust their strategies to make Player II indifferent between her pure strategies, and so on.

Assume $0 < A < C < B$. We now determine the cut points A, B, and C by solving three indifference equations as follows.

1. For Player I to be indifferent at A:

1. If Player I checks at $x = A$, his expected payoff is

$$\int_0^A\int_0^A 2dz\,dy + \int_0^A\int_A^1 -1dz\,dy + \int_A^1\int_0^1 -1dz\,dy.$$

2. If Player I bets at $x = A$, his expected payoff is

$$\int_0^C\int_0^C 2dz\,dy + \int_0^C\int_C^1 -(b+1)dz\,dy + \int_C^1\int_0^1 -(b+1)dz\,dy.$$

Equating the two expressions above yields the following equation:

$$3A^2 - 1 = 3C^2 + bC^2 - b - 1.$$

2. For Player I to be indifferent at B:

1. If Player I checks at $x = B$, his expected payoff is

$$\int_0^B\int_0^B 2dz\,dy + \int_0^B\int_B^1 -1dz\,dy + \int_B^1\int_0^1 -1dz\,dy.$$

2. If Player I bets at $x = B$, his expected payoff is

$$\int_0^C\int_0^C 2dz\,dy + \int_0^C\int_C^B (b+2)dz\,dy + \int_C^B\int_0^C (b+2)dz\,dy$$
$$+ \int_C^B\int_C^B 2(b+1)dz\,dy$$
$$+ \int_B^1\int_0^B -(b+1)dz\,dy + \int_0^B\int_B^1 -(b+1)dz\,dy$$
$$+ \int_B^1\int_B^1 -(b+1)dz\,dy.$$

Equating the two expressions above yields the following equation:

$$3B^2 - 1 = -2bCB + 3bB^2 + 3B^2 - b - 1.$$

3. For Player II (or Player III) to be indifferent at C: Assuming Player I bets:

1. If Player II folds at $y = C$, her expected payoff is

$$\int_0^A \int_0^1 -1 dz\, dx + \int_B^1 \int_0^1 -1 dz\, dx.$$

2. If Player II calls at $y = C$, her expected payoff is

$$\int_0^A \int_0^C (b+2) dz\, dx + \int_0^A \int_C^1 -(b+1) dz\, dx + \int_B^1 \int_0^1 -(b+1) dz\, dx.$$

Equating the two expressions above yields the following equation:

$$-A + B - 1 = 2bCA + 3CA - bA - A - b + bB + B - 1.$$

Solving the above nonlinear system of three equations in three unknowns gives us the solutions for A, B, and C for the Nash equilibrium strategies.

In particular, when $b = 2$ the optimal cuts are

$$A = 0.137058194328370,$$
$$B = 0.829422249795391,$$
$$C = 0.641304115985175.$$

This results in the value of the game (for Player I) being 0.122557074714865.

We can also determine the best bet amount b that maximizes Player I's payoff under the Nash equilibrium strategies. Approximately, $b^* \approx 2.07$, resulting in Player I achieving a maximum payoff of 0.122590664136184. Therefore, we observe that the highest payoff for Player I in the three-player game exceeds that of von Neumann's two-player game, which is $1/9 = 0.111111\ldots$, achieved at $b^* = 2$.

One final remark is that the Nash equilibrium for the three-player continuous game resembles those observed in the discrete model when n is large. In our experiments with the finite-deck model, we are able to simulate up to $n = 65$. With $b = 2$, we obtain the following results:

- The cuts are

$$A = (8 + 14/23)/65 = 0.132441471571906,$$
$$B = 1 - 11/65 = 0.830769230769231,$$
$$C = 1 - (22 + 189/205)/65 = 0.647354596622889.$$

- The value of the game (for Player I) is $974/8121 = 0.119935968476789$.

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