On The Limiting Distributions of the Total Height On Families of Trees

By Andrew LOHR and Doron ZEILBERGER

First Version, For Historical Purposes Only; Please use Current Version

Abstract: A symbolic-computational algorithm, fully implemented in Maple, is described, that computes explicit expressions for generating functions that enable the efficient computations of the expectation, variance, and higher moments, of the random variable 'sum of distances to the root', defined on any given family of rooted ordered trees (defined by degree restrictions). We provide convincing evidence that the limiting (scaled) distributions are all the same, and coincide with the limiting distribution of the same random variable, when it is defined on labeled rooted trees, thereby introducing a (hopefully) new universality class of combinatorial random variables.

Maple packages and Sample Input and Output Files

This article is accompanied by Maple packages, `TREES.txt`, and `THS.txt`, and several input and output files available from the front of this article

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/otrees.html` .

Background

While many natural families of combinatorial random variables, X_n, indexed by a positive integer n, (for example, tossing a coin n times and noting the number of Heads, or counting the number of occurrences of a specific pattern in an n-permutation) have different expectations, μ_n, and different standard deviations, σ_n, and (usually) largely different asymptotic expressions for these, yet the centralized and scaled versions, $Z_n := \frac{X_n-\mu_n}{\sigma_n}$, very often, converge (in distribution) to the standard normal distribution whose probability density function is famously $\frac{1}{\sqrt{2\pi}} exp(-\frac{x^2}{2})$, and whose moments are $0, 1, 0, 3, 0, 5, 0, 15, 0, 105, \dots$. Such sequences of random variables are called *asymptotically normal*. Whenever this is **not** the case, it is a cause for excitement. One celebrated case (see Dan Romik's [Ro] masterpiece for an engaging and detailed description) is the random variable 'largest increasing subsequence', defined on the set of permutations, where the intriguing *Tracy-Widom distribution* shows up.

Other, more recent, examples of *abnormal* limiting distributions are described in [Z1], [EZ1],[EZ2], and [EZ3].

In this article, we continue in the same vein as in [EZ2]. In that article, the random variable 'sum of the distances from the root', defined on the set of *labelled rooted trees* on n vertices, was considered, and it was shown how to find explicit expressions for any given moment, and the first 12 moments were derived, extending the pioneering work of John Riordan and Neil Sloane ([RiS]), who derived an explicit formula for the expectation. In [EZ2], one of us (DZ) pledged a donation of 100 dollars to the OEIS Foundation in honor of the first to identify the centralized-scaled limiting distribution. The exact and approximate values for α_3 (the *skewness*), α_4 (the *kurtosis*), and the higher moments through the ninth turn out to be as follows.

$$\alpha_3 = \frac{\left(6\,\pi - \frac{75}{4}\right)\sqrt{3}\sqrt{\frac{\pi}{10-3\,\pi}}}{10-3\,\pi} = 0.7005665293596503\ldots \quad ,$$

$$\alpha_4 = \frac{-189\,\pi^2 + 315\,\pi + 884}{7\,(10-3\,\pi)^2} = 3.560394897132889\ldots \quad ,$$

$$\alpha_5 = \frac{\left(36\,\pi^2 + \frac{75}{2}\,\pi - \frac{105845}{224}\right)\sqrt{3}\sqrt{\frac{\pi}{10-3\,\pi}}}{(10-3\,\pi)^2} = 7.2563753582799571\ldots \quad ,$$

$$\alpha_6 = \frac{15}{16016}\,\frac{-144144\,\pi^3 - 720720\,\pi^2 + 3013725\,\pi + 2120320}{(10-3\,\pi)^3} = 27.685525695770609\ldots \quad ,$$

$$\alpha_7 = \frac{\left(162\,\pi^3 + \frac{6615}{4}\,\pi^2 - \frac{103965}{32}\,\pi - \frac{101897475}{9152}\right)\sqrt{3}\sqrt{\frac{\pi}{10-3\,\pi}}}{(10-3\,\pi)^3} = 90.0171829093603301\ldots \quad ,$$

$$\alpha_8 = \frac{3}{2586584}\,\frac{-488864376\,\pi^4 - 8147739600\,\pi^3 - 455885430\,\pi^2 + 86568885375\,\pi + 32820007040}{(10-3\,\pi)^4}$$

$$= 358.80904151261251\ldots \quad ,$$

$$\alpha_9 = \frac{\left(648\,\pi^4 + 15795\,\pi^3 + \frac{591867}{16}\,\pi^2 - \frac{461286225}{2288}\,\pi - \frac{188411947088175}{662165504}\right)\sqrt{3}\sqrt{\frac{\pi}{10-3\,\pi}}}{(10-3\,\pi)^4} = 1460.7011342971821\ldots \quad .$$

This Article

In this article we extend the work of [EZ2] and treat infinitely many other families of trees. For any given set of positive integers, S, we will have a 'sample space' of all ordered rooted trees where a vertex may have no children (i.e. be a *leaf*) or it **must** have a number of children that belongs to S. If $S = \{2\}$ we have the case of *complete binary trees.*

For each such family, defined by S, we will show how to derive explicit expressions for the generating functions of the numerators of the straight moments, from which one can easily get many values, and very efficiently find the numerical values for the moments-about-the-mean and hence the scaled moments. For the special case of complete binary trees, we will derive explicit expressions for the first nine moments (that may be extended indefinitely), as well as explicit expressions for the asymptotics of the scaled moments, and surprise! they coincide *exactly* with those found in [EZ2] for the case of labelled rooted trees. This leads us to conjecture that the limiting distribution is the *same* for each such family. Hence, another 'universality' class is born!

When we googled the kurtosis (α_4), $3.560394897132\ldots$, to see whether anyone else came up with it before, we found *one* hit, our own [EZ3], where the random variable was the number of inversions in 132-avoiding permutations. In hindsight it is not surprising, since both 132-avoiding permutations and complete binary trees are counted by Catalan numbers, and there is, possibly, a natural bijection between these objects that sends the total height to the number of inversions. What seemed *surprising* (to us) is that it seems to coincide with the case of labelled rooted trees treated in [EZ2].

Rooted Ordered Trees

Recall that an *ordered rooted tree* is an unlabeled graph with the root drawn at the top, and each vertex has a certain number (possibly zero) of children, drawn from left to right. For any finite set of positive integers, S, let $\mathcal{T}(S)$ be the set of all rooted labelled trees where each vertex either has no children, or else has a number of children that belongs to S. The set $\mathcal{T}(S)$ has the following structure ("grammar")

$$\mathcal{T}(S) = \{\cdot\} \bigcup_{i \in S} \{\cdot\} \times \mathcal{T}(S)^i \quad .$$

Fix S, Let f_n be number of rooted ordered trees in $\mathcal{T}(S)$ with exactly n vertices. It follows immediately, by elementary generatingfunctionology, that the ordinary generating function

$$f(x) := \sum_{n=0}^{\infty} f_n \, x^n \quad ,$$

(that is the sum of the weights of *all* members of $\mathcal{T}(S)$ with the weight $x^{NumberOfVertices}$ assigned to each tree) satisfies the **algebraic** equation

$$f(x) = x \left(1 + \sum_{i \in S} f(x)^i \right) \quad .$$

Given an ordered tree, t, define the random variable $H(t)$ to be the sum of the distances to the root of all vertices. Let H_n be its restriction to the subset of $\mathcal{T}(S)$, let's call it $\mathcal{T}_n(S)$, of members of $\mathcal{T}(S)$ with exactly n vertices. Our goal in this article is to describe a symbolic-computational algorithm that, for *any* finite set S of positive integers, *automatically* finds generating functions that enable the fast computation of the average, variance, and as many higher moments as desired. We will be particularly interested in the limit, as $n \to \infty$, of the centralized-scaled distribution, and we have strong evidence to conjecture that it is always the same as the one for rooted labelled trees found in [EZ2].

Let $P_n(y)$ be the generating polynomial defined over $\mathcal{T}_n(S)$, of the random variable, 'sum of distances from the root'. Define the *grand generating function*

$$F(x, y) = \sum_{n=0}^{\infty} P_n(y) x^n \quad .$$

Consider a typical tree, t, in $\mathcal{T}_n(S)$, and now define the more general *weight* by $x^{NumberOfVertices} \, y^{H(t)} = x^n \, y^{H(t)}$. If t is a singleton, then its weight is simply $x^1 y^0 = x$, but if its sub-trees (the trees whose roots are the children of the original root) are $t_1, t_2, \ldots t_i$ (where $i \in S$), then

$$H(t) = H(t_1) + \ldots + H(t_i) + n - 1 \quad ,$$

since when you make the tree t, out of subtrees $t_1, \ldots, t_i$ by placing them from left to right and then attaching them to the root, each vertex gets its 'distance to the root' increased by 1, so altogether the sum of the vertices' heights gets increased by the total number of vertices in $t_1, \ldots, t_i$ (i.e. $n-1$). Hence $F(x,y)$ satisfies the **functional equation**

$$F(x,y) = x \cdot \left(1 + \sum_{i \in S} F(xy,y)^i\right) \quad ,$$

that can be used to generate many terms of the sequence of generating polynomials $\{P_n(y)\}$.

Note that when $y=1$, $F(x,1) = f(x)$, and we get back the algebraic equation satisfied by $f(x)$.

From Enumeration to Statistics in General

Suppose that we have a finite set, A, on which a certain numerical attribute, called *random variable*, X, (using the probability/statistics lingo), is defined.

For any non-negative integer i, let's define

$$N_i := \sum_{a \in A} X(a)^i \quad .$$

In particular, $N_0(X)$ is the number of elements of A.

The expectation of X, $E[X]$, denoted by μ, is, of course,

$$\mu \; = \; \frac{N_1}{N_0} \quad .$$

For $i > 1$, the i-th straight moment is

$$E[X^i] \; = \; \frac{N_i}{N_0} \quad .$$

The i-th *moment about the mean* is

$$m_i := E[(X-\mu)^i] = E[\sum_{r=0}^{i} \binom{i}{r} (-1)^r \mu^r X^{i-r}] = \sum_{r=0}^{i} (-1)^r \binom{i}{r} \mu^r E[X^{i-r}]$$

$$= \sum_{r=0}^{i} (-1)^r \binom{i}{r} \left(\frac{N_1}{N_0}\right)^r \frac{N_{i-r}}{N_0}$$

$$= \frac{1}{N_0^i} \sum_{r=0}^{i} (-1)^r \binom{i}{r} N_1^r N_0^{i-r-1} N_{i-r} \quad .$$

Finally, the most interesting quantities, statistically speaking, apart from the mean μ and variance m_2 are the **scaled-moments**, also known as, *alpha coefficients*, defined by

$$\alpha_i := \frac{m_i}{m_2^{i/2}} \quad .$$

Using Generating functions

In our case X is H_n (the sum of the vertices' distances to the root, defined over rooted ordered trees in our family, with n vertices), and we have

$$N_1(n) = P_n'(1)$$

$$N_i(n) = (y\frac{d}{dy})^i P_n(y)\big|_{y=1} .$$

It is more convenient to first find the numerators of the factorial moments

$$F_i(n) = (\frac{d}{dy})^i P_n(y)|_{y=1} \quad ,$$

from which $N_i(n)$ can be easily found, using the Stirling numbers of the second kind.

Automatic Generation of Generating functions for the (Numerators of the) Factorial Moments

Let's define

$$P(X) = 1 + \sum_{i \in S} X^i \quad ,$$

then our functional equation for the grand-generating function, $F(x, y)$ can be written

$$F(x, y) = xP(F(xy, y)) \quad .$$

If we want to get generating functions for the first k factorial moments of our random variable H_n, we need the first k coefficients of the Taylor expansion, about $y = 1$, of $F(x, y)$. Writing $y = 1 + z$, and

$$G(x, z) = F(x, 1 + z) \quad ,$$

we get the functional equation for $G(x, z)$

$$G(x, z) = x\, P(G(x + xz, z)) \quad . \qquad (FE)$$

Let's write the Taylor expansion of $G(x, z)$ around $z = 0$ to order k

$$G(x, z) = \sum_{r=0}^{k} g_r(x) \frac{z^r}{r!} + O(z^{k+1}) \quad .$$

It follows that

$$G(x+xz,z) = \sum_{r=0}^{k} g_r(x+xz)\frac{z^r}{r!} + O(z^{k+1}) \quad .$$

We now do the Taylor expansion of $g_r(x+xz)$ around x, getting

$$g_r(x+xz) = g_r(x) \,+\, g_r'(x)(xz) \,+\, g_r''(x)\frac{(xz)^2}{2!} \,+\, \ldots \,+\, g_r^{(k)}(x)\frac{(xz)^k}{k!} \,+\, O(z^{k+1}) \quad .$$

Plugging all this into (FE), and comparing coefficients of respective terms of z^r for r from 0 to k we get $k+1$ extremely complicated equations relating $g_r^{(j)}(x)$ to each other. It is easy to see that one can express $g_r(x)$ in terms of $g_s^{(j)}(x)$ with $s < r$ (and $0 \le j \le k$) .

Using implicit differentiation, the derivatives of $g_0(x)$, $g_0^{(j)}(x)$ (where $g_0(x)$ is the same as $f(x)$), can be expressed as rational functions of x and $g_0(x)$. As soon as we get an expression for $g_r(x)$ in terms of x and $g_0(x)$, we can use calculus to get expressions for the derivatives $g_r^{(j)}(x)$ in terms of x and $g_0(x)$. At the end of the day, we get expressions for each $g_r(x)$ in terms of x and $g_0(x)$ (alias $f(x)$), and since it is easy to find the first ten thousand (or whatever) Taylor coefficients of $g_0(x)$, we can get the first ten thousand coefficients of $g_r(x)$, for all $0 \le r \le k$, and get the numerical sequences that will enable us to get the above-mentioned statistical information.

The beauty is that this is all done by the computer! Maple knows calculus.

We can do even better. Using the methods described in [FS], one should be able to get, *automatically*, asymptotic formulas for the expectation, variance, and as many moments as desired. Alas, implementing it in general would have to wait for the future.

For the special case of complete binary trees, everything can be expressed in terms of Catalan numbers, and hence the asymptotic is easy, and our beloved computer, running the Maple package `TREES.txt` (mentioned above), obtained the results in the next section.

Computer-Generated Theorems About the Expectation, Variance, and First Nine Moments for the Total Height on Complete Binary Trees on n Leaves

See the output file

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oTREES3.txt` .

Universality

The computer output, given in the above webpage, proved that for this case, of complete binary trees, the limits of the first nine scaled moments coincide *exactly* with those found in [EZ2], and given above. Numerical evidence, obtained thanks to the closed-form expressions for the generating functions for the numerators of the moments obtained by our package, indicates that for randomly-chosen sets, S, the random variable H_n defined on $\mathcal{T}_n(S)$ always has the same limiting distribution.

We conjecture that this continuous probability distribution, whose probability density function is yet to be found, and for which we know the exact values of the first twelve moments, is *universal* in the sense of applying to all families of ordered trees considered here.

Conclusion

Even more interesting than the *actual* research reported here, it the way that is was obtained. Fully automatically!

References

[EZ1] Shalosh B. Ekhad and Doron Zeilberger, *Explicit Expressions for the Variance and Higher Moments of the Size of a Simultaneous Core Partition and its Limiting Distribution* , The Personal Journal of Shalosh B. Ekhad and Doron Zeilberger,
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/stcore.html` .

[EZ2] Shalosh B. Ekhad and Doron Zeilberger, *Going Back to Neil Sloane's FIRST LOVE (OEIS Sequence A435): On the Total Heights in Rooted Labeled Trees*, The Personal Journal of Shalosh B. Ekhad and Doron Zeilberger,
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/a435.html` .

[EZ3] Shalosh B. Ekhad and Doron Zeilberger, *Automatic Proofs of Asymptotic ABNORMALITY (and much more!) of Natural Statistics Defined on Catalan-Counted Combinatorial Families*, The Personal Journal of Shalosh B. Ekhad and Doron Zeilberger,
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/abnormal.html` .

[FS] Philippe Flajolet Robert Sedgewick, "*Analytic Combinatorics*", Cambridge University Press, 2009. (Free download from Flajolet's homepage.)

[RiS] John Riordan and Neil J. A. Sloane, *The enumeration of rooted trees by total height*, J. Australian Math. Soc. **10**(1969), 278-282. Available on line from:
`http://neilsloane.com/doc/riordan-enum-trees-by-height.pdf` .

[Ro] Dan Romik, "*The Surprising Mathematics of Longest Increasing Subsequences*", Cambridge University Press, 2015.

[Z1] Doron Zeilberger, *Doron Gepner's Statistics on Words in* $\{1,2,3\}^*$ *is (Most Probably) Asymptotically Logistic*, The Personal Journal of Shalosh B. Ekhad and Doron Zeilberger,
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/gepner.html` .

Andrew Lohr, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA. ajl213 at math dot rutgers dot edu; `http://www.math.rutgers.edu/~ajl213/` .

Doron Zeilberger, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.
DoronZeil at gmail dot com ; `http://www.math.rutgers.edu/~zeilberg/` .

Written: Feb. 9, 2017.