Arab. J. Math.
https://doi.org/10.1007/s40065-025-00524-0

Arabian Journal of Mathematics

Shalosh B. Ekhad · Doron Zeilberger

Efficient weighted counting of multiset derangements

Dedicated to Mourad El-Houssieny ISMAIL (born April 27, 1944), with friendship and admiration

Received: 24 April 2025 / Accepted: 29 April 2025

Abstract We use the Almkvist–Zeilberger algorithm, combined with a weighted version of the Even–Gillis Laguerre integral due to Foata and Zeilberger, in order to efficiently compute weight enumerators of multiset derangements according to the number of cycles. The present paper is inspired by important previous work by Mourad Ismail and his collaborators, done in the late 1970s, but still useful after all these years.

Mathematics Subject Classification (2000) 05A

Multiset derangements

The (generalized) Laguerre polynomials, $L_k^{(\alpha)}(x)$, are defined as follows:

$$L_k^{(\alpha)}(x) := \sum_{i=0}^{n}(-1)^i \frac{(\alpha+i+1)(\alpha+i+2)\dots(\alpha+n)}{i!(n-i)!} x^i \quad .$$

Let $A_1, A_2, \dots, A_n$ be n pairwise-disjoint sets of cardinalities $k_1, \dots, k_n$ respectively. A permutation π of $A := A_1 \cup \dots \cup A_n$ is a multiset derangement if for every $1 \le i \le n$, whenever $x \in A_i$, $\pi(x) \notin A_i$. Let $D(k_1, \dots, k_n)$ be the set of such multiset derangements.

As usual, for any finite set S, let $|S|$ denote its number of elements.

Even and Gillis [7] (see also [2,9–11]) proved that

$$|D(k_1, \dots, k_n)| = (-1)^{k_1+\dots+k_n} \left(\prod_{i=1}^{n} k_i!\right) \int_0^\infty \left(\prod_{i=1}^{n} L_{k_i}^{(0)}(x)\right) e^{-x}\, dx \quad . \tag{1}$$

Comment: Usually the elements of each A_i are identified, and the formula then does not have $\left(\prod_{i=1}^{n} k_i!\right)$ in front, but for the purpose of the present paper, as was done in [8], we have $k_1 + \dots + k_n$ distinct elements, all preserving their individuality.

For a permutation π, let $cyc(\pi)$ denote its *number of cycles*. For example $cyc(1234) = 4$, and $cyc(2341) = 1$. Recall that, famously,

$$\sum_{\pi \in S_n} \alpha^{cyc(\pi)} = \alpha(\alpha+1)\cdots(\alpha+n-1) \quad .$$

S. B. Ekhad · D. Zeilberger (✉)
Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA. E-mail: DoronZeil@gmail.com

S. B. Ekhad
E-mail: ShaloshBEkhad@gmail.com

Published online: 18 May 2025

This is easily proved by considering how to form permutations of $\{1,\ldots,n\}$ out of those of $\{1,\ldots,n-1\}$. Denote the left side by $f_n(\alpha)$. Given a permutation of $\{1,\ldots,n-1\}$ we can insert n inside any of the existing cycles, and there are $n-1$ ways of doing it, preserving the number of cycles, or else create a brand-new cycle with n alone, increasing the number of cycles by one. Hence $f_n(\alpha)=((n-1)+\alpha)f_{n-1}(\alpha)$.

Let $A(k_1,\ldots,k_n)(\alpha)$ be the weight enumerator, according to the number of cycles, of the set of multiset derangements, $D(k_1,\ldots,k_n)$, in other words

$$A(k_1,\ldots,k_n)(\alpha) := \sum_{\pi\in D(k_1,\ldots,k_n)} \alpha^{cyc(\pi)} \quad .$$

In 1988, Dominique Foata and one of us (DZ) [8], inspired by the work of Mourad Ismail and his collaborators [2–4] proved the following α-analog of the Even-Gillis theorem:

$$A(k_1,\ldots,k_n)(\alpha) = \frac{(-1)^{k_1+\cdots+k_n}}{(\alpha-1)!}\left(\prod_{i=1}^{n} k_i!\right)\int_0^\infty \left(\prod_{i=1}^{n} L_{k_i}^{(\alpha-1)}(x)\right) x^{\alpha-1}e^{-x}\,dx \quad . \tag{2}$$

Note that this is a polynomial of degree $(k_1+\cdots+k_n)/2$ rather than $k_1+\cdots+k_n$, since every cycle is at least of length 2.

In this paper we will focus on efficient computations of many terms of the sequences
$A(k,k,\ldots,k)(\alpha)$, where k is repeated n times, for specific (small, and not so small) k, but arbitrarily large n. In other words, our goal is to compute as many as possible terms of the sequences

$$F_k(n)(\alpha) := A(k,\ldots,k)(\alpha) \quad , \quad (k \quad \text{repeated} \quad n \quad \text{times}) \quad ,$$

for $k=1$, $k=2$, etc.

By (2) we have

$$F_k(n)(\alpha) = \frac{(-1)^{kn}k!^n}{(\alpha-1)!}\int_0^\infty (L_k^{(\alpha-1)}(x))^n x^{\alpha-1}e^{-x}\,dx \quad . \tag{3}$$

Using the **Almkvist–Zeilberger algorithm** [1] (see [5] for a lucid and engaging account), one of us (SBE), using our Maple package `Mourad.txt`, available from the front of this article

https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/mourad.html,

discovered (and proved) the following recurrences for $F_k(n) = F_k(n)(\alpha)$ for $1 \le k \le 10$. They get increasingly complicated, and we will only state the first two in the body of this paper.

$$-\alpha\,(n+1)\,F_1(n) - (n+1)\,F_1(n+1) + F_1(n+2) = 0 \quad .$$

$$\begin{aligned}&4\alpha\,(2n+5)\,(n+2)\,(n+1)\,(\alpha+1)^2\,F_2(n)\\&+2\,(n+2)\,(\alpha+1)\left(4\alpha\,n^2+12\alpha n-4n^2+7\alpha-14n-10\right)F_2(n+1)\\&-2\,(n+2)\left(4\alpha n+4n^2+8\alpha+16n+17\right)F_2(n+2)+(2n+3)\,F_2(n+3)=0 \quad .\end{aligned} \tag{5}$$

For linear recurrences for $F_k(n)$ for $3 \le k \le 10$ see the output file

https://sites.math.rutgers.edu/~zeilberg/tokhniot/oMourad1.txt.

We observe that the order of the recurrence for $F_k(n)$ is $k+1$ and its degree in n is $\binom{k+1}{2}$.

These recurrences enable very fast computation of quite a few terms of these sequences. It is all implemented in the already mentioned Maple package `Mourad.txt`. The direct url of this package is:

https://sites.math.rutgers.edu/~zeilberg/tokhniot/Mourad.txt.

A user's manual to the maple package Mourad.txt

To use it first download it to your favorite directory (usually Downloads). Start a Maple worksheet, make sure that the directory is the right one, and then type

```
read `Mourad.txt` ;  .
```

To get a list of the procedures, type `ezra();`. The main procedures are:

- `Wder(L,a)`, that inputs a list of positive integers, $L = [k_1, \ldots, k_n]$ and implements Eq. (2). This is very slow, and should not be used for large L.
- `Operk(k,a,n,N)`, that inputs a positive integer k, and outputs the recurrence operator (where N is the shift operator in n) annihilating the sequence $F_k(n)$. These get very complicated for human eyes, but the computer does not mind and it enables a fast computation of many terms.
- `SeqkF(k,K,α)`: inputs a positive integer k and a positive integer K and a symbol α and outputs the first K terms of the sequence of polynomials $F_k(n) = F_k(n)(\alpha)$. For example, to get the weight enumerator of the set of permutations of a standard deck of cards where no card can wind up at a location that originally was occupied by the same number (1 through 13, where Jack, Queen, and King stand for 11, 12, 13 respectively) but it is OK to have the same suit, type:

  `SeqkF(4,13,α)[13];` ,

  and you would get, in a few seconds, the following polynomial in α of degree 26

$$
\begin{aligned}
&626486325682388256883179081695232\,\alpha^{26}\\
&\quad+3948815860811007759557670403206807552\,\alpha^{25}\\
&\quad+4226160446928101410675933447042193424384\,\alpha^{24}\\
&\quad+1829313185198525509532452983498671376039936\,\alpha^{23}\\
&\quad+425955227133577312273392421310068029118218240\,\alpha^{22}\\
&\quad+61568711382255715699343414832865761752795578368\,\alpha^{21}\\
&\quad+6015599331237497842549834616372527226200006852608\,\alpha^{20}\\
&\quad+420030513102996289545618495318355347968579239673856\,\alpha^{19}\\
&\quad+21779385529606788308065066752435641655566027030790144\,\alpha^{18}\\
&\quad+861931009463580565142515454351924475556603802576486400\,\alpha^{17}\\
&\quad+26556926811772603306934511893782498309330811792400580608\,\alpha^{16}\\
&\quad+646219419386602045907824228576682527851206056554484727808\,\alpha^{15}\\
&\quad+12544166147808400841334081628081554018739662394272604225536\,\alpha^{14}\\
&\quad+195525408546538912690378251287680488219792943092212919435264\,\alpha^{13}\\
&\quad+2455695605166443718371007842011087818790955115435503879454720\,\alpha^{12}\\
&\quad+24867048146672227309345666989913796704728810126752820020379648\,\alpha^{11}\\
&\quad+202569793911613274182929019082185092261014157201085369153486848\,\alpha^{10}\\
&\quad+1320388339665569428585764027609539765653334771119656423470923776\,\alpha^{9}\\
&\quad+6825167923093955037138102373992833000704975000443998456569135104\,\alpha^{8}\\
&\quad+27602809328921835313793682068121303712142304270099611821308641280\,\alpha^{7}\\
&\quad+85647342705993322148148235777401007447932223607159691210985046016\,\alpha^{6}\\
&\quad+198159663830900044042641789039253865122617020230065397080602443776\,\alpha^{5}\\
&\quad+327547473685724687587188995032714624999306890307177019801201541112\,\alpha^{4}\\
&\quad+361148215004517312493645900517444844168859774724070502768740139008\,\alpha^{3}\\
&\quad+234426065400514976953417524798811902707109969381695319447196139520\,\alpha^{2}\\
&\quad+66394948050946830932484058263644488672722608355067055619597926400\,\alpha \quad .
\end{aligned}
$$

Plugging-in $\alpha = 1$ and dividing by $4!^{13}$ (to get derangements where members of the same set are identified) gives you:

$$
14938044444990933549162842901889480312298804695556 \quad ,
$$

agreeing with the title of [6], that handled the *straight*, rather than the *weighted*, enumeration of multiset derangements.

Note that the largest coefficient in the above polynomial is that of α^3, hence the *mode* of the random variable 'number of cycles' in a random multi-set derangement of a standard deck is 3. The average happens to be $3.586337835\ldots$ while the standard-deviation (the squre-root of the variance) is $1.412546929\ldots$

Acknowledgements Many thanks to the two anonymous referees for very helpful suggestions and corrections on the first draft.

Author Contributions All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by Shalosh B. Ekhad and Doron Zeilberger. The first draft of the manuscript was written by Doron Zeilberger, and all authors commented on previous versions of the manuscript.

Funding The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

Data availability not applicable.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

References

1. Almkvist, G.; Zeilberger, D.: The method of differentiating under the integral sign. J. Symb. Comput. **10**, 571–591 (1990). https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/duis.html
2. Askey, R.: Orthogonal Polynomials and Special Functions. SIAM (1975)
3. Askey, R.; Ismail, M.; Koornwinder, T.: Weighted permutation problems and Laguerre polynomials. J. Comb. Theory (Ser. A.) **25**, 277–287 (1978)
4. Askey, R.; Ismail, M.E.H.; Rashed, M.T.: A derangement problem. Technical report 1522, UW-Madison Mathematics Research Center, June 1975
5. Dougherty-Bliss, R.: Integral recurrences from A to Z. Am. Math. Mon. **129**, 805–815 (2021)
6. Ekhad, S.B.; Koutschan, C.; Zeilberger, D.: There are EXACTLY 1493804444499093354916284290188948031229880469556 ways to derange a standard deck of cards (ignoring suits) [and many other such useful facts]. Enumerative Combinatorics and Applications, vol. 1, issue 3, article S2A17 (2021). https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/multider.html
7. Even, S.; Gillis, J.: Derangements and Laguerre polynomials. Math. Proc. Camb. Philos. Soc. **79**, 135–143 (1976)
8. Foata, D.; Zeilberger, D.: Laguerre polynomials, weighted derangements, and positivity. SIAM J. Discrete Math. **1**, 425–433 (1988)
9. Gillis, J.: The statistics of derangement—a survey. J. Stat. Phys. **58**, 575–578 (1990)
10. Gillis, J.; Ismail, M.; Offer, T.: An asymptotic problem in derangement theory. SIAM J. Math. Anal. **21**, 262–269 (1990)
11. Zeilberger, D.: How Joe Gillis discovered combinatorial special function theory. Math. Intell. **17**(2), 65–66 (1995). https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/gillis.html