USING THE "FRESHMAN'S DREAM" IDENTITY TO PROVE COMBINATORIAL CONGRUENCES

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Abstract. In a recent beautiful but technical article, William Y.C. Chen, Qing-Hu Hou, and Doron Zeilberger developed an algorithms for finding congruence, mod p, of sequences of indefinite sums of many combinatorial sequences, namely those (like the Catalan and Motzkin sequences) that are expressible in terms of constant terms of powers of Laurent polynomials. Here we extend it in two directions. The Laurent polynomials in question may be of several variables, and instead of single sums we have multiple sums. In fact we even combine these two generalizations.

Introduction

In a recent elegant article, [2], the following type of quantities were considered

$$\left(\sum_{k=0}^{rp-1} a(k)\right) \mod p \ ,$$

where $a(k)$ is a combinatorial sequence, expressible as the constant term of a power of a Laurent polynomial of a **singel** variable, k; r is a positive integer; and p is a *general* (symbolic) prime.

Their method, while ingenious, is very elementary! All they need is the

The Freshman's Dream Identity

$$(a+b)^p \equiv_p a^p + b^p \quad ,$$

where $x \equiv_p y$ means $x \equiv y \mod p$.

Recall that the easy proof follows from using the Binomial theorem, and noting that $\binom{p}{r}$ is divisible by p except when $r = 0$ and $r = p$. This also leads, to one of the many proofs of the grandmother of all congruences, **Fermat's Little Theorem**, $a^p \equiv_p a$, by starting with $0^p \equiv_p 0$, and applying induction to $(a+1)^p \equiv_p a^p + 1^p$.

The second ingredient in the ingenious method of [2] is even more elementary

Sum of a Geometric Series:

$$\sum_{i=0}^{n-1} z^i = \frac{1-z^n}{1-z} \quad ,$$

The focus in the Chen-Hou-Zeilberger paper was both computer-algebra implementation, and proving a general theorem about a wide class of sums, and it is rather technical, and hence its beauty is lost to a wider audience. The first purpose of the present article is to give a more leisurly introduction to their method, and illustrate it with some illuminating examples. The main purpose, however, is to extend it in **two** directions. The summand $a(k)$, may be the constant term of a Laurent polynomial in **several** variables, and instead of a **single** summation sign, we can have *multi-sums*. In fact we can combine these two

Notation The constant term of a Laurent polynomial $P(x_1, x_2, \ldots, x_n)$, alias the coefficient of $x_1^0 x_2^0 .. x_n^0$, is denoted by $CT[P(x_1, x_2, \ldots, x_n)]$ and the general coefficient of $x_1^{n_1} x_2^{n_2} .. x_n^{n_k}$ in $P(x_1, x_2, \ldots, x_n)$ is denoted by $COEFF_{[x_1^{n_1} x_2^{n_2} .. x_n^{n_k}]} P(x_1, x_2, \ldots, x_n)$. For example,

$$CT\left[\frac{1}{xy} + 3 + 5xy - x^3 + 6y^2\right] = 3 \text{ and } COEFF_{[xy]}\left[\frac{1}{xy} + 3 + 5xy + x^3 + 6y^2\right] = 5.$$

We use the symmetric representation of integers in $[-|m|/2, |m|/2]$ when reducing modulo integer m. For example, $6 \bmod 5 = 1$ and $4 \bmod 5 = -1$.

Review of the Chen-Hou-Zeilberger Single Variable Case

In order to motivate our generalization, we will first review, in more detail then given in [2], some of their elegant results.

Proposition 1. For any prime p and positive integer r, define

$$A(r;p) := \sum_{n=0}^{rp-1} \binom{2n}{n}. \text{ Then, } A(1;p) \bmod p = \begin{cases} 1, & \text{if } p \equiv 1 \mod 3 \\ -1, & \text{if } p \equiv 2 \mod 3 \end{cases}.$$

Proof: Using the fact that $\binom{2n}{n} = CT\left[\frac{(1+x)^{2n}}{x^n}\right]$ and $(a+b)^p \equiv_p a^p + b^p$, we have

$$
\begin{aligned}
\sum_{n=0}^{p-1}\binom{2n}{n} &= \sum_{n=0}^{p-1} CT\left[\left(\frac{(1+x)^{2n}}{x^n}\right)\right] \\
&= \sum_{n=0}^{p-1} CT\left[\left(2+x+\frac{1}{x}\right)^n\right] \\
&= CT\left[\frac{\left(2+x+\frac{1}{x}\right)^p-1}{2+x+\frac{1}{x}-1}\right] \\
&\equiv_p CT\left[\frac{2^p+x^p+\frac{1}{x^p}-1}{1+x+\frac{1}{x}}\right] \text{ By Freshman's dream} \\
&\equiv_p CT\left[\frac{2+x^p+\frac{1}{x^p}-1}{1+x+\frac{1}{x}}\right] \text{ By Fermat's little theorem} \\
&= CT\left[\frac{1+x^p+\frac{1}{x^p}}{1+x+\frac{1}{x}}\right] \\
&= CT\left[\frac{1+x^p+x^{2p}}{(1+x+x^2)x^{p-1}}\right] \\
&= COEFF_{[x^{p-1}]}\left[\frac{1}{1+x+x^2}\right] \\
&= COEFF_{[x^{p-1}]}\left[\frac{1-x}{1-x^3}\right]. \\
&= COEFF_{[x^p]}\sum_{i=0}^{\infty}x^{3i+1}-\sum_{i=0}^{\infty}x^{3i+2}.
\end{aligned}
$$

The result follows from extracting the coefficient of x^p from the two sums on the right side.

Corollary 1 (Corollary 2.3, [2]): $A(2;p) \bmod p = \begin{cases} 3, & \text{if } p \equiv 1 \mod 3 \\ -3, & \text{if } p \equiv 2 \mod 3 \end{cases}.$

Proof: Proceeding as in Theorem 1 above,

$$\begin{aligned}
\sum_{n=0}^{2p-1}\binom{2n}{n} &= \sum_{n=0}^{2p-1} CT\left[\left(\frac{(1+x)^{2n}}{x^n}\right)\right] \\
&= CT\left[\frac{\left(2+x+\frac{1}{x}\right)^{2p}-1}{2+x+\frac{1}{x}-1}\right] \\
&= CT\left[\frac{\left(6+4x+\frac{4}{x}+x^2+\frac{1}{x^2}\right)^{p}-1}{2+x+\frac{1}{x}-1}\right] \\
&\equiv_p CT\left[\frac{\left(6+4x^p+\frac{4}{x^p}+x^{2p}+\frac{1}{x^{2p}}\right)-1}{2+x+\frac{1}{x}-1}\right] \\
&\equiv_p COEFF_{[x^{2p-1}]}\left[\frac{1+4x^p}{1+x+x^2}\right] \\
&\equiv_p COEFF_{[x^{2p-1}]}\left[\frac{1}{1+x+x^2}\right] + 4COEFF_{[x^{p-1}]}\left[\frac{1}{1+x+x^2}\right] .
\end{aligned}$$

The result follows from Theorem 1 and the last congruence.

Corollary 2 (Catalan Numbers): Let $a(n)$ be the constant term of $(1-x)\left(2+x+\dfrac{1}{x}\right)^n$ and let

$$A(r;p) := \sum_{n=0}^{rp-1} a(n) .$$

Then, $A(1;p) \bmod p = \begin{cases} 1, & \text{if } p \equiv 1 \mod 3 \\ -2, & \text{if } p \equiv 2 \mod 3 . \end{cases}$

Proof: Continuing as above,

$$\begin{aligned}
\sum_{n=0}^{p-1} a(n) &= \sum_{n=0}^{p-1} CT\left[(1-x)\left(2+x+\frac{1}{x}\right)^n\right] \\
&= CT\left[\frac{(1-x)\left(\left(2+x+\frac{1}{x}\right)^p-1\right)}{2+x+\frac{1}{x}-1}\right] \\
&\equiv_p CT\left[\frac{(1-x)\left(\left(2+x^p+\frac{1}{x^p}\right)-1\right)}{2+x+\frac{1}{x}-1}\right] \\
&\equiv_p COEFF_{[x^{p-1}]}\left[\frac{1-x}{1+x+x^2}\right] \\
&\equiv_p COEFF_{[x^{p-1}]}\left[\frac{1}{1+x+x^2}\right] - COEFF_{[x^{p-2}]}\left[\frac{1}{1+x+x^2}\right] .
\end{aligned}$$

The result follows from Theorem 1 and the last congruence.

Theorem 2 (Motzkin Numbers). Let $a(n)$ be the constant term of $(1-x^2)\left(1+x+\frac{1}{x}\right)^n$, and define

$$A(r;p) := \sum_{n=0}^{rp-1} a(n) \ .$$

Then, $A(1;p) \bmod p \ = \begin{cases} 2, & \text{if } p \ \equiv \ 1 \mod 4 \\ -2, & \text{if } p \ \equiv \ 3 \mod 4 \ . \end{cases}$

Proof:

$$\begin{aligned} \sum_{n=0}^{p-1} a(n) &= \sum_{n=0}^{p-1} CT\left[(1-x^2)\left(1+x+\frac{1}{x}\right)^n\right] \\ &= CT\left[\frac{(1-x^2)\left((1+x+\frac{1}{x})^p-1\right)}{1+x+\frac{1}{x}-1}\right] \\ &\equiv_p COEFF_{[x^{p-1}]}\left[\frac{1-x^2}{1+x^2}\right] \\ &\equiv_p COEFF_{[x^{p-1}]}\left[\frac{1}{1+x^2}\right] - COEFF_{[x^{p-3}]}\left[\frac{1}{1+x^2}\right] \end{aligned}$$

The result follows from series expansion of $\frac{1}{1+x^2}$ and the last congruence.

Corollary 3: Let $a(n)$ be the constant term of $(1-x^2)\left(1+x+\frac{1}{x}\right)^n$ and let

$$A(r;p) := \sum_{n=0}^{rp-1} a(n)\,.$$

Then, $A(2;p) \bmod p \ = \begin{cases} 4, & \text{if } p \ \equiv \ 1 \mod 4 \\ -4, & \text{if } p \ \equiv \ 3 \mod 4 \ . \end{cases}$

Proof:

$$\begin{aligned}
\sum_{n=0}^{2p-1} a(n) &= \sum_{n=0}^{2p-1} CT\left[(1-x^2)\left(1+x+\frac{1}{x}\right)^n\right] \\
&= CT\left[\frac{(1-x^2)\left(\left(1+x+\frac{1}{x}\right)^{2p}-1\right)}{1+x+\frac{1}{x}-1}\right] \\
&= CT\left[\frac{(1-x^2)\left(\left(3+2x+x^2+\frac{2}{x}+\frac{1}{x^2}\right)^{p}-1\right)}{1+x+\frac{1}{x}-1}\right] \\
&= CT\left[\frac{(1-x^2)\left(\left(3+2x^p+x^{2p}+\frac{2}{x^p}+\frac{1}{x^{2p}}\right)-1\right)}{1+x+\frac{1}{x}-1}\right] \\
&\equiv_p COEFF_{[x^{2p-1}]}\left[\frac{(1-x^2)(1+2x^p)}{1+x^2}\right] \\
&\equiv_p COEFF_{[x^{2p-1}]}\left[\frac{1}{1+x^2}\right] + 2COEFF_{[x^{p-1}]}\left[\frac{1}{1+x^2}\right] \\
&- COEFF_{[x^{2p-3}]}\left[\frac{1}{1+x^2}\right] - 2COEFF_{[x^{p-3}]}\left[\frac{1}{1+x^2}\right].
\end{aligned}$$

The result follows from Theorem 2 and the last congruence.

From the above theorems and corollaries, it is easy to observe that partial sums with upper summation limit of the form $rp-1$, for $r>1$, can always be expressed in terms of the sum with upper summation limit $p-1$. This observation leads us to the following simplification of Theorem 2.1 in [2].

Theorem 3 . Let $P(x)$ be a Laurent polynomial in x and let p be a prime. Let $R(x)$ be the denominator, after clearing, of the expression

$$\frac{P(x^p)-1}{P(x)-1}.$$

Then, for any positive integer r and Laurent polynomial $Q(x)$,

$$\left(\sum_{n=0}^{rp-1} CT\left[P(x)^n Q(x)\right]\right) \bmod p\ ,$$

is congruent to a finite linear combination of finite terms of coefficients of the rational function $\dfrac{1}{R(x)}$.

Multi-Sums and Multi-Variables

Theorem 4. Let p be a prime number and let r and s be a positive integers. Let

$$A(r,s;p) := \sum_{n=0}^{rp-1}\sum_{m=0}^{sp-1}\binom{n+m}{m}^2 .$$

Then, $A(1,1;p) \bmod p = \begin{cases} 0, & \text{if } p \equiv 0 \mod 3 \\ 1, & \text{if } p \equiv 1 \mod 3 \\ -1, & \text{if } p \equiv 2 \mod 3 \end{cases}$.

Furthermore, $A(2,2;p) \equiv_p COEFF_{[x^{2p-1}y^{2p-1}]} \dfrac{1+4x^p+4y^p+16x^py^p}{(1+x+xy)(1+y+xy)}$.

Proof: Let $P(x,y) = (1+y)\left(1+\dfrac{1}{x}\right)$ and $Q(x,y) = (1+x)\left(1+\dfrac{1}{y}\right)$. First observe that

$$\binom{n+m}{m}^2 = \binom{n+m}{m}\binom{n+m}{n} = CT\left(P(x,y)^n Q(x,y)^m\right) .$$

Then,

$$\begin{aligned}
\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\binom{m+n}{m}^2 &= \sum_{m=0}^{p-1}\sum_{n=0}^{p-1} CT\left[P(x,y)^n Q(x,y)^m\right] \\
&= \sum_{m=0}^{p-1} CT\left[\frac{P(x,y)^p-1)Q(x,y)^m}{P(x,y)-1}\right] \\
&= CT\left[\left(\frac{P(x,y)^p-1}{P(x,y)-1}\right)\left(\frac{Q(x,y)^p-1}{Q(x,y)-1}\right)\right]
\end{aligned}$$

Using $(a+b)^p \equiv (a^p+b^p) \bmod p$, we can pass to mod p as above

$$\begin{aligned}
\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\binom{m+n}{m}^2 &\equiv_p CT\left[\left(\frac{P(x^p,y^p)-1}{P(x,y)-1}\right)\left(\frac{Q(x^p,y^p)-1}{Q(x,y)-1}\right)\right] \\
&\equiv_p CT\left[\frac{(1+y^p+x^py^p)(1+x^p+x^py^p)}{(1+y+xy)(1+x+xy)x^{p-1}y^{p-1}}\right] \\
&\equiv_p COEFF_{[x^{p-1}y^{p-1}]}\left[\frac{(1+y^p+x^py^p)(1+x^p+x^py^p)}{(1+y+xy)(1+x+xy)}\right] \\
&\equiv_p COEFF_{[x^{p-1}y^{p-1}]}\left[\frac{1}{(1+y+xy)(1+x+xy)}\right]
\end{aligned}$$

Using the Apagodu-Zeilberger algorithm ([1]), the diagonal coefficients satisfy the recurrence equation $N^2+N+1=0$ with initial conditions $a(0)=1, a(1)=1, a(2)=0$. The result now follows from the fact that this recurrence is equivalent to $N^3-1=0$ and the solution to this recurrence is given by:

$$a(n) = \begin{cases} 1, & \text{if } n \equiv 0 \mod 3 \\ 1, & \text{if } n \equiv 1 \mod 3 \\ 0, & \text{if } n \equiv 2 \mod 3 \end{cases} .$$

Note that our partial sum is congruent to $a(p-1)$, not $a(p)$.

For the $A(2,2;p)$, we have

$$\begin{aligned}\sum_{m=0}^{2p-1}\sum_{n=0}^{2p-1}\binom{m+n}{m}^2 &\equiv_p CT\left[\left(\frac{P(x^p,y^p)^2-1}{P(x,y)-1}\right)\left(\frac{Q(x^p,y^p)^2-1}{Q(x,y)-1}\right)\right]\\ &\equiv_p CT\left[\frac{(1+y^p+x^py^p)^2(1+x^p+x^py^p)^2}{(1+y+xy)(1+x+xy)x^{p-1}y^{p-1}}\right]\\ &\equiv_p COEFF_{[x^{2p-1}y^{2p-1}]}\left[\frac{(1+y^p+x^py^p)^2(1+x^p+x^py^p)^2}{(1+y+xy)(1+x+xy)}\right]\\ &\equiv_p COEFF_{[x^{2p-1}y^{2p-1}]}\left[\frac{1+4x^p+4y^p+16x^py^p}{(1+x+xy)(1+y+xy)}\right]\end{aligned}$$

It follows that,

$$A(2,2;p)\equiv_p COEFF_{[x^{2p-1}y^{2p-1}]}\frac{1+4x^p+4y^p+16x^py^p}{(1+x+xy)(1+y+xy)}\ .$$

By symmetry, it simplifies to

$$A(2,2;p)\equiv_p COEFF_{[x^{2p-1}y^{2p-1}]}\frac{1}{(1+x+xy)(1+y+xy)}+$$

$$8COEFF_{[x^{p-1}y^{2p-1}]}\frac{1}{(1+x+xy)(1+y+xy)}+16COEFF_{[x^{p-1}y^{p-1}]}\frac{1}{(1+x+xy)(1+y+xy)}$$

Based on computer calculations, we conjecture other values of r and s that admits a nice form. These congruence are also true mod p^2. The congruences with higher power of p are called supercongruence and the present method can't handle them.

(1) For $p\geq 17,\ A(2,2;p) \bmod p = \begin{cases} 7, & \text{if } p \equiv 1 \mod 3\\ -7, & \text{if } p \equiv 2 \mod 3\end{cases}.$
(2) For $p\geq 37,\ A(2,3;p) \bmod p = \begin{cases} 17, & \text{if } p \equiv 1 \mod 3\\ -17, & \text{if } p \equiv 2 \mod 3\end{cases}.$
(3) For $p\geq 127,\ A(3,3;p) \bmod p = \begin{cases} 63, & \text{if } p \equiv 1 \mod 3\\ -63, & \text{if } p \equiv 2 \mod 3\end{cases}.$
(4) $A(1,1;p)\equiv 1 \pmod{p^2}$.
(5) For $p\geq 5,\ A(2,2;p) \bmod p^2 = \begin{cases} 7, & \text{if } p \equiv 1 \mod 3\\ -7, & \text{if } p \equiv 2 \mod 3\end{cases}.$
(6) For $p\geq 7,\ A(2,3;p) \bmod p^2 = \begin{cases} 17, & \text{if } p \equiv 1 \mod 3\\ -17, & \text{if } p \equiv 2 \mod 3\end{cases}.$
(7) For $p\geq 17,\ A(3,3;p) \bmod p^2 = \begin{cases} 63, & \text{if } p \equiv 1 \mod 3\\ -63, & \text{if } p \equiv 2 \mod 3\end{cases}.$

Remark: If we take the third power of the summand in Theorem 4 and define

$$C(r,s;p):=\sum_{n=0}^{rp-1}\sum_{m=0}^{sp-1}\binom{n+m}{m}^3,$$

then,

$$\binom{n+m}{m}^3 = CT\left[\frac{(1+x)^{m+n}}{x^m}\frac{(1+y)^{m+n}}{y^m}\frac{(1+z)^{m+n}}{z^n}\right] .$$

with $P(x,y,z) = \left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)(1+z)$ and $Q(x,y,z) = (1+x)(1+y)\left(1+\frac{1}{z}\right)$, we have

$$\binom{n+m}{m}^3 = CT\left((P(x,y)^m Q(x,y)^n\right) .$$

and proceeding as above, we get

$$C(1,1;p) \equiv COEFF_{[x^{p-1}y^{p-1}z^{p-1}]}\left[\frac{1}{p(x,y,z)q(x,y,z)}\right] \bmod p\,,$$

where $p(x,y,z) = 1+x+y+xy+xz+yz+xyz$ and $q(x,y,z) = 1+y+z+xy+xz+yz+xyz$.

Theorem 5. Let $p>2$ be prime, and r,s and t be positive integers. Define

$$A(r,s,t;p) := \sum_{m_1=0}^{rp-1}\sum_{m_2=0}^{sp-1}\sum_{m_3=0}^{tp-1}\binom{m_1+m_2+m_3}{m_1,m_2,m_3} .$$

Then, $A(1,1,1;p) \equiv 1 \bmod p\,.$

Proof: First observe that $\binom{m_1+m_2+m_3}{m_1,m_2,m_3} = CT\left[\frac{(x+y+z)^{m_1+m_2+m_3}}{x^{m_1}y^{m_2}z^{m_3}}\right]$. Hence

$$\begin{aligned}
\sum_{0\le m_1,m_2,m_3\le p-1}\binom{m_1+m_2+m_3}{m_1,m_2,m_3} &= \sum_{0\le m_1,m_2,m_3\le p-1} CT[(x+y+z)^(m_1+m_2+m_3)/(x^{m_1}y^{m_2}z^{m_3})]\\
&= CT\left[\sum_{0\le m_1,m_2,m_3\le p-1}\frac{(x+y+z)^(m_1+m_2+m_3)}{x^{m_1}y^{m_2}z^{m_3}}\right]\\
&= CT[\sum_{0\le m_1,m_2,m_3\le p-1}\frac{(x+y+z)^{m_1}}{x^{m_1}}\frac{(x+y+z)^{m_2}}{y^{m_2}}\frac{(x+y+z)^{m_3}}{y^{m_3}}\\
&= CT[\left(\sum_{0\le m_1\le p-1}\left(\frac{x+y+z}{x}\right)^{m_1}\right)\left(\sum_{0\le m_2\le p-1}\left(\frac{x+y+z}{y}\right)^{m_2}\right)\\
&\quad \left(\sum_{0\le m_3\le p-1}\left(\frac{x+y+z}{z}\right)^{m_3}\right)\\
&= CT\left[\frac{(\frac{x+y+z}{x})^p-1}{\frac{x+y+z}{x}-1}\times\frac{(\frac{x+y+z}{y})^p-1}{\frac{x+y+z}{y}-1}\times\frac{(\frac{x+y+z}{z})^p-1}{\frac{x+y+z}{z}-1}\right]\\
&= COEFF_{[x^{p-1}y^{p-1}z^{p-1}]}[\frac{(x+y+z)^p-x^p}{y+z}\times\frac{(x+y+z)^p-y^p}{x+z}\times\\
&\quad \frac{(x+y+z)^p-z^p}{x+y}].
\end{aligned}$$

So far this is true for all p, not only p prime. Now take it mod p and get, since $(x+y+z)^p \equiv_p (x^p+y^p+z^p)$,

$$\begin{aligned}
A(1,1,1;p) &= COEFF_{[x^{p-1}y^{p-1}z^{p-1}]}\left(\frac{y^p+z^p}{y+z}\times\frac{x^p+z^p}{x+z}\times\frac{y^p+z^p}{y+z}\right)\\
&= COEFF_{[x^{p-1}y^{p-1}z^{p-1}]}\left(y^{p-1}z^0-y^{p-2}z+y^{p-3}z^2+...+y^0z^{p-1}\right)\times\\
&\quad \left(x^{p-1}z^0-x^{p-2}z+x^{p-3}z^2+...+x^0z^{p-1}\right)\times\left(x^{p-1}y^0-x^{p-2}y+x^{p-3}y^2+...+x^0y^{p-1}\right)\\
&= COEFF_{[x^{p-1}y^{p-1}z^{p-1}]}[\left(\sum_{j=0}^{p-1}(-1)^jx^{p-1-j}y^j\right)\left(\sum_{k=0}^{p-1}(-1)^{k+1}z^{p-1-k}x^k\right)\left(\sum_{l=0}^{p-1}(-1)^ly^{p-1-l}z^l\right)].
\end{aligned}$$

It is easy to see that to extract the $x^{p-1}y^{p-1}z^{p-1}$ term in the above triple sum, we need $j=k=l=(p-1)/2$, and hence get $(-1)^{3*(p-1)/2}$ expansion by multiplying $(x^{p-1-j}y^j)(z^{p-1-k}x^k)(y^{p-1-l}z^l)$ exactly when $j=k=l=(p-1)/2$. Hence the coefficient of $x^{p-1}y^{p-1}z^{p-1}$ is
$\sum_{j=0}^{p-1}(-1)^{3j}=\sum_{j=0}^{p-1}(-1)^j$. It follows that the coefficient is 1 for $p>2$ and 0 for $p=2$. This completes the proof.

We make the following conjectures based on computer calculations.

(1) $A(1,1,2;p)\equiv 2 \pmod p$, for $p\ge 5$.
(2) $A(1,2,2;p)\equiv 5 \pmod p$, for $p\ge 11$.

(3) $A(2,2,2;p) \equiv 16 \pmod{p}$. for $p \geq 37$.

The following supercongruences are also true.

(1) $A(1,1,1;p) \equiv 1 \pmod{p^2}$, for $p \geq 2$.
(2) $A(1,1,1;p) \equiv 1 \pmod{p^3}$, for $p \geq 2$.
(3) $A(1,1,2;p) \equiv 2 \pmod{p^2}$, for $p \geq 3$.
(4) $A(1,1,2;p) \equiv 2 \pmod{p^3}$, for $p \geq 3$.
(5) $A(1,2,2;p) \equiv 5 \pmod{p^2}$, for $p \geq 5$.
(6) $A(1,2,2;p) \equiv 5 \pmod{p^3}$, for $p \geq 3$.
(7) $A(2,2,2;p) \equiv 16 \pmod{p^2}$, for $p \geq 7$.
(8) $A(2,2,2;p) \equiv 16 \pmod{p^3}$, for $p \geq 5$.

Similarly, if we define

$$A(r,s,t,u;p) := \sum_{m_1=0}^{rp-1} \sum_{m_2=0}^{sp-1} \sum_{m_3=0}^{tp-1} \sum_{m_4=0}^{up-1} \binom{m_1+m_2+m_3+m_4}{m_1,m_2,m_3,m_4} ,$$

for prime p and positive integers r, s, t and u, then we get the following conjectures.

(1) $A(1,1,1,1;p) \equiv 1 \pmod{p}$, for $p \geq 3$.
(2) $A(1,1,1,2;p) \equiv 2 \pmod{p}$, for $p \geq 5$.
(3) $A(1,1,2,2;p) \equiv 5 \pmod{p}$, for $p \geq 11$.
(4) $A(1,2,2,2;p) \equiv 16 \pmod{p}$, for $p \geq 37$.

Our last observation is the general multinomial case. For prime $p > 2$ and positive integers $r_i, i = 1,2,3,\ldots,k$, if we define

$$A(r_1,r_2,\ldots,r_k;p) := \sum_{m_1=0}^{r_1p-1} \sum_{m_2=0}^{r_2p-1} \cdots \sum_{m_a=0}^{r_kp-1} \binom{m_1+m_2\ldots+m_a}{m_1,m_2,\ldots,m_k} .$$

Then, computer outputs support the following conjectures

(1) $A(1,1,\ldots,1;p) = 1 \pmod{p}$, for $p \geq 2$.
(2) $A(1,1,\ldots,1;p) = 1 \pmod{p^2}$, for $p \geq 2$.
(3) $A(1,1,\ldots,1;p) = 1 \pmod{p^3}$, for $p \geq 2$.

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