A SHORT PROOF OF MCDOUGALL'S CIRCLE THEOREM

MARC CHAMBERLAND AND DORON ZEILBERGER

In fond memory of Andrei Zelevinsky (1953-2013)
who loved Ptomeny's Theorem.

Ptolemy's Theorem is a beautiful, classical result concerning quadrilaterals. Specifically, let P_1, P_2, P_3, P_4 be cyclically ordered points on a circle and $d_{i,j}$ denote the distance between P_i and P_j. Then Ptolemy's Theorem states $d_{1,3}d_{2,4} = d_{1,2}d_{3,4} + d_{1,4}d_{2,3}$. Refinements known as the Brahmagupta-Mahavira identities [1] lead to a "ratio version" of Ptolemy's Theorem:

$$(1) \qquad \frac{d_{1,3}}{d_{2,4}} = \frac{d_{1,2}\, d_{1,4} + d_{2,3}\, d_{3,4}}{d_{1,4}\, d_{3,4} + d_{1,2}\, d_{2,3}}.$$

Equation (1) can be written as

$$\frac{1}{d_{1,2}\, d_{1,3}\, d_{1,4}} + \frac{1}{d_{1,2}\, d_{2,3}\, d_{2,4}} = \frac{1}{d_{1,3}\, d_{2,3}\, d_{3,4}} + \frac{1}{d_{1,4}\, d_{2,4}\, d_{3,4}}.$$

Jane McDougall [2] has generalized this result from 4 points to $2n$ points.

Theorem 1. *(McDougall) Let n be a positive integer and P_i, $1 \leq i \leq 2n$, be cyclically ordered points on a circle. Furthermore, let*

$$R_i := \prod_{\substack{1 \leq j \leq 2n \\ j \neq i}} d_{i,j}.$$

Then

$$(2) \qquad \sum_{i=1}^{n} \frac{1}{R_{2i}} = \sum_{i=1}^{n} \frac{1}{R_{2i-1}}.$$

McDougall's proof, using tools from complex analysis, follows by applying harmonic mappings to a class of minimal surfaces. Related use of these methods can be found in [3]. The goal of this note is to provide a short, elementary proof of McDougall's theorem. The key to the proof is using the *Lagrange Interpolation Formula*: If $P(z)$ is a polynomial whose degree does not exceed $N - 1$, then

$$P(z) = \sum_{i=1}^{N} \frac{(z - z_1) \cdots (z - z_{i-1})(z - z_{i+1}) \cdots (z - z_N)}{(z_i - z_1) \cdots (z_i - z_{i-1})(z_i - z_{i+1}) \cdots (z_i - z_N)} P(z_i)$$

for any distinct numbers $z_1, \ldots, z_N$. Using a formula like this is not surprising since Equation (2) involves sums of products.

Proof. Without loss of generality, assume the circle is centered at the origin and has radius one. Denote the points as $P_i = (\cos 2t_i, \sin 2t_i)$, $1 \le i \le 2n$, where $0 \le t_1 < t_2 < \cdots < t_{2n} < \pi$. Basic trigonometry can be used to show that $d_{i,j} = 2\sin(t_i - t_j)$. Letting $u_i = e^{It_i}$ where $I = \sqrt{-1}$, it follows that

$$d_{i,j} = -I\left(\frac{u_i^2 - u_j^2}{u_i u_j}\right)$$

whenever $i < j$. This produces

$$I^{2n-1}(-1)^{i-1}\frac{1}{R_i} = \left(\prod_{j=1}^{2n} u_j\right)\frac{u_i^{2n-2}}{\prod_{j\neq i}(u_i^2 - u_j^2)}.$$

With Equation (2) in view, we construct

$$(3) \qquad I^{2n-1}\sum_{i=1}^{2n}(-1)^{i-1}\frac{1}{R_i} = \left(\prod_{j=1}^{2n} u_j\right)\sum_{i=1}^{2n}\frac{u_i^{2n-2}}{\prod_{j\neq i}(u_i^2 - u_j^2)}.$$

Applying the Lagrange Interpolation Formula with $P(z) = z^r$, where $r < N - 1$, the coefficient of z^{N-1} yields

$$0 = \sum_{i=1}^{N}\frac{z_i^r}{(z_i - z_1)\cdots(z_i - z_{i-1})(z_i - z_{i+1})\cdots(z_i - z_N)}.$$

Taking $N = 2n$, $r = n - 1$ and $z_i = u_i^2$ for $i = 1, 2, \cdots, 2n$, the z_i terms are distinct and the right side of Equation (3) must equal zero. This is equivalent to the desired result. $\qquad\square$

Remark 1. The Lagrange Interpolation Formula is easy to prove, so no sophisticated machinery is needed for proving McDougall's Theorem.

Remark 2. Letting the circle's radius approach infinity gives the corrolary that Equation (2) holds if the $2n$ points are collinear. In fact, the equation still holds on lines even if the number of points is odd:

$$\sum_{i=1}^{2n-1}\frac{(-1)^i}{R_i} = 0.$$

We leave this as an exercise for the reader.

<div align="center">REFERENCES</div>

[1] Askey, R. *Completing Brahmagupta's extension of Ptolemy's theorem*, The legacy of Alladi Ramakrishnan in the mathematical sciences, 191–197, Springer, New York, 2010.

[2] McDougall, J. private communication, 2012.

[3] McDougall, J. *Harmonic Mappings with Quadrilateral Image*, Complex analysis and potential theory, 99–115, CRM Proc. Lecture Notes, **55**, Amer. Math. Soc., Providence, RI, 2012.

Department of Mathematics and Statistics, Grinnell College, Grinnell, IA 50112.

E-mail address: chamberl@math.grinnell.edu

Mathematics Department, Rutgers University (New Brunswick), Piscataway, NJ 08854.

E-mail address: zeilberg@math.rutgers.edu