# A Short Proof of McDougall's Circle Theorem

Marc Chamberland and Doron Zeilberger

**Abstract**

This note offers a short, elementary proof of a result similar to Ptolemy's theorem. Specifically, let $d_{i,j}$ denote the distance between $P_i$ and $P_j$. Let $n$ be a positive integer and $P_i$, for $1 \le i \le 2n$, be cyclically ordered points on a circle. If

$$R_i := \prod_{\substack{1 \le j \le 2n \\ j \ne i}} d_{i,j},$$

then

$$\sum_{i=1}^{n} \frac{1}{R_{2i}} = \sum_{i=1}^{n} \frac{1}{R_{2i-1}}. \tag{1}$$


*In fond memory of Andrei Zelevinsky (1953-2013)*
*who loved Ptolemy's Theorem.*

Ptolemy's Theorem is a beautiful, classical result concerning quadrilaterals. Specifically, let $P_1, P_2, P_3$, and $P_4$ be cyclically ordered points on a circle and $d_{i,j}$ denote the distance between $P_i$ and $P_j$. Then Ptolemy's Theorem states that $d_{1,3}d_{2,4} = d_{1,2}d_{3,4} + d_{1,4}d_{2,3}$. Refinements, known as the Brahmagupta-Mahavira identities [1], lead to a "ratio version" of Ptolemy's Theorem:

$$\frac{d_{1,3}}{d_{2,4}} = \frac{d_{1,2}\ d_{1,4} + d_{2,3}\ d_{3,4}}{d_{1,4}\ d_{3,4} + d_{1,2}\ d_{2,3}}. \tag{2}$$

Equation (2) can be written as

$$\frac{1}{d_{1,2}\ d_{1,3}\ d_{1,4}} + \frac{1}{d_{3,1}\ d_{3,2}\ d_{3,4}} = \frac{1}{d_{2,1}\ d_{2,3}\ d_{2,4}} + \frac{1}{d_{4,1}\ d_{4,2}\ d_{4,3}}.$$

Jane McDougall [2] has generalized this result from 4 points to $2n$ points.

**Theorem 1.1.** *(McDougall) Let $n$ be a positive integer and $P_i$, for $1 \le i \le 2n$, be cyclically ordered points on a circle. If*

$$R_i := \prod_{\substack{1 \le j \le 2n \\ j \ne i}} d_{i,j},$$

*then*

$$\sum_{i=1}^{n} \frac{1}{R_{2i}} = \sum_{i=1}^{n} \frac{1}{R_{2i-1}}. \tag{3}$$

McDougall's proof, using tools from complex analysis, follows by applying harmonic mappings to a class of minimal surfaces. Related use of these methods can be found in [3]. The goal of this note is to provide a short, elementary proof of McDougall's theorem. The key to the proof is using the *Lagrange Interpolation Formula*: If $P(z)$ is a polynomial whose degree does not exceed $N-1$, then

$$P(z) = \sum_{i=1}^{N} \frac{(z-z_1)\cdots(z-z_{i-1})(z-z_{i+1})\cdots(z-z_N)}{(z_i-z_1)\cdots(z_i-z_{i-1})(z_i-z_{i+1})\cdots(z_i-z_N)} P(z_i)$$

for any distinct numbers $z_1, \ldots, z_N$. Using a formula like this is not surprising since Equation (3) involves sums of products.

*Proof.* Without loss of generality, assume the circle is centered at the origin and has radius one. Denote the points as $P_i = (\cos 2t_i, \sin 2t_i)$, for $1 \le i \le 2n$, where $0 \le t_1 < t_2 < \cdots < t_{2n} < \pi$. Basic trigonometry can be used to show that $d_{i,j} = 2\sin(t_i - t_j)$. Letting $u_i = e^{It_i}$ where $I = \sqrt{-1}$, it follows that

$$d_{i,j} = -I\left(\frac{u_i^2 - u_j^2}{u_i u_j}\right)$$

whenever $i < j$. This produces

$$I^{2n-1}(-1)^{i-1}\frac{1}{R_i} = \left(\prod_{j=1}^{2n} u_j\right)\frac{u_i^{2n-2}}{\prod_{j\neq i}(u_i^2 - u_j^2)}.$$

With Equation (3) in view, we construct

$$I^{2n-1}\sum_{i=1}^{2n}(-1)^{i-1}\frac{1}{R_i} = \left(\prod_{j=1}^{2n} u_j\right)\sum_{i=1}^{2n}\frac{u_i^{2n-2}}{\prod_{j\neq i}(u_i^2 - u_j^2)}. \tag{4}$$

Proving the theorem is therefore equivalent to proving that the sum on the right side of Equation (4) equals zero.

Applying the Lagrange Interpolation Formula with $P(z) = z^r$, where $r < N-1$, the coefficient of $z^{N-1}$ yields

$$0 = \sum_{i=1}^{N} \frac{z_i^r}{(z_i-z_1)\cdots(z_i-z_{i-1})(z_i-z_{i+1})\cdots(z_i-z_N)}. \tag{5}$$

Taking $N = 2n$, $r = n-1$ and $z_i = u_i^2$ for $i = 1, 2, \cdots, 2n$, the $z_i$ terms are distinct and Equation (5) becomes

$$0 = \sum_{i=1}^{2n} \frac{u_i^{2n-2}}{\prod_{j\neq i}(u_i^2 - u_j^2)}.$$

This proves the theorem. $\square$

**Remark 1**. The Lagrange Interpolation Formula is easy to prove; the left and right sides agree when $z = z_i$, for $1 \leq i \leq N$, thus the two degree $N - 1$ polynomials must agree everywhere. Therefore, no sophisticated machinery is needed to prove McDougall's Theorem.

**Remark 2**. Letting the circle's radius approach infinity gives the corollary that Equation (3) holds if the $2n$ points are collinear. In fact, the equation still holds on lines even if the number of points is odd:

$$\sum_{i=1}^{2n-1} \frac{(-1)^i}{R_i} = 0.$$

We leave this as an exercise for the reader.

## References


[1] R. Askey, Completing Brahmagupta's extension of Ptolemy's theorem, in *The legacy of Alladi Ramakrishnan in the mathematical sciences*, Edited by Krishnaswami Alladi, John R. Klauder and Calyampudi R. Rao, Springer, New York, 2010. 191–197,

[2] J. McDougall, private communication, 2012.

[3] J. McDougall, Harmonic Mappings with Quadrilateral Image, in *Complex analysis and potential theory*, CRM Proc. Lecture Notes, Vol. 55, American Mathematical Society, Providence, 2012. 99–115.



*Department of Mathematics and Statistics, Grinnell College, Grinnell, IA 50112*
*chamberl@math.grinnell.edu*

*Department of Mathematics, Rutgers University (New Brunswick), Piscataway, NJ 08854*
*zeilberg@math.rutgers.edu*