The Gift Exchange Problem, Part II

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So far we had a k-free (partial) recurrence for $E_\sigma(n,k)$ (for $\sigma = 2,3$) (that implied recurrences for G_σ(n)) and the trivial n-free (partial) recurrence, (5), for the general $E_\sigma(n,k)$, It turns out that $E_\sigma(n,k)$ satisfies a pure (ordinary) recurrence in k, and that $G_\sigma(n)$ satisfies a linear recurrence with polynomial coefficients, in n, for *every* σ.

Indeed: we have

$$\sum_k E_\sigma(n,k)\frac{y^k}{k!} = \frac{1}{n!}\left(\sum_{i=0}^{\sigma+1}\frac{y^i}{i!}\right)^n$$

Using the J.C.P.Miller ([Z1]) method we get the *pure recurrence* that enables us, for a given n, to compute fast the numbers $E_\sigma(n,k)$ for a *fixed* n, without having to use the $E_\sigma(n-1,k)$'s

$$\sum_{i=0}^{\sigma}\left(\frac{n}{i!} - \frac{k}{(i+1)!} + \frac{1}{(i-1)!+i!}\right) E_\sigma(n,k-i) \ = 0 \qquad .$$

We also have, using Euler's integral representation for the factorial

$$k! = \int_0^\infty e^{-y} y^k \, dy \quad ,$$

that we have the integral representation

$$G_\sigma(n) = \frac{1}{n!}\int_0^\infty e^{-y}\left(\sum_{i=0}^{\sigma+1}\frac{y^i}{i!}\right)^n \, dy.$$

It follows from the (discrete) Almkvist-Zeilberger algorithm [AlZ] that $G_\sigma(n)$ satisfies *some* homogeneous linear recurrence with polynomial (in n) coefficients , and it follows from Theorem AZ of [ApZ] that its order is $\sigma+1$. The Maple package `DavidNeil` accompanying this article can easily compute these recurrences, and the the recurrences for $1 \le \sigma \le 6$, together with the implied asymptotics (using the Poincare-Birkhoff-Trijinksi method as exposited in [WiZ] and implemented in [Z2]) can be found in:

`http://www.math.rutgers.edu/~ zeilberg/tokhniot/oDavidNeil1` .

Regarding the *generating function*

$$g_\sigma(x) := \sum_{n=0}^{\infty} G_\sigma(n) x^n,$$

We have,

$$g_\sigma(x) = \int_0^\infty \exp(x\left(-y + \sum_{i=0}^{\sigma+1}\frac{y^i}{i!}\right))\, dy.$$

it follows from the (continuous) Almkvist-Zeilberger algorithm [AlZ] that $g_\sigma(x)$ satisfies *some* homogeneous linear differential equation with polynomial (in x) coefficients , and it follows from the obvious analog of Theorem AZ of [ApZ] that its order is $\sigma + 1$.

See

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oDavidNeil2`

for the differential equations for $1 \leq \sigma \leq 6$.

These differential equations transcribe to linear recurrences in $G_\sigma(n)$, that are *not* minimal-order, and for small σ coincide with the ones obtained above using Sister Celine's technique.

Additional References

[AlZ] G. Almkvist and D. Zeilberger, The method of differentiating under the integral sign, J. Symbolic Computation 10 (1990), 571-591.

[ApZ] Moa Apagodu and Doron Zeilberger, *Multi-Variable Zeilberger and Almkvist-Zeilberger Algorithms and the Sharpening of Wilf-Zeilberger Theory*, Adv. Appl. Math. **37**(2006), 139-152.

[WiZ] Jet Wimp and Doron Zeilberger, *Resurrecting the asymptotics of linear recurrences*, J. Math. Anal. Appl. **111**(1985), 162-177.

[Z1] Doron Zeilbeger *The J.C.P. Miller Recurrence for exponentiating a polynomial, and its q-Analog*, J. Difference Eqs. and Appls. **1**(1995), 57-60.

[Z2] Doron Zeilbeger, *AsyRec: A Maple package for Computing the Asymptotics of Solutions of Linear Recurrence Equations with Polynomial Coefficients*, The Personal Journal of Ekhad and Zeilberger, `http://www.math.rutgers.edu/ zeilberg/pj.html`