THE MAHONIAN PROBABILITY DISTRIBUTION ON WORDS IS ASYMPTOTICALLY NORMAL

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ABSTRACT. We study XXX

[Abstract! Correct date! No showkeys!] xxx

1. INTRODUCTION

S:intro

The most important discrete probability distribution, by far, is the *Binomial distribution*, $B(n,p)$ for which we know everything *explicitly*, $\mathbb{P}(X=i)$ $(\binom{n}{i}p^i(1-p)^{n-i})$, the probability generating function $((pt+(1-p))^n)$, the moment generating function $((pe^t+1-p)^n)$, etc. etc. Most importantly, it is *asymptotically normal*, which means that the normalized random variable

$$Z_n = \frac{X_n - np}{\sqrt{np(1-p)}}$$

tends to the standard Normal distribution $N(0,1)$, as $n \to \infty$.

Another important discrete distribution function is the *Mahonian* distribution, defined on the set of *permutations* on n objects, and describing, inter-alia, the random variable "number of inversions". (Recall that an inversion in a permutation $\pi_1,\dots,\pi_n$ is a pair $1 \le i < j \le n$ such that $\pi_i > \pi_j$). Let us call this random variable M_n. The probability generating function, due to Netto, is given *explicitly* by:

$$F_n(q) = \frac{1}{n!}\prod_{i=1}^{n}\frac{1-q^i}{1-q} \quad . \tag{1.1}$$ netto

The formula (1.1) has a simple probabilistic interpretation (see Feller's account in [3, Section X.6]): If Y_j is the number of i with $1 \le i < j$ and $\pi_i > \pi_j$, then

$$M_n = Y_1 + \dots + Y_n, \tag{1.2}$$ nettoy

and $Y_1,\dots,Y_n$ are independent random variables and Y_j is uniformly distributed on $\{0,\dots,j-1\}$, as is easily seen by constructing π by inserting

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Accompanied by Maple package MahonianStat available from
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/mahon.html`. The work of D. Zeilberger was supported in part by the United States of America National Science Foundation.

$1, \dots, n$ in this order at random positions; thus Y_j has probability generating function $(1-q^j)/(j(1-q))$. It follows from (1.1) or (1.2) by simple calculations that the Mahonian distribution has mean and variance

$$\mathbb{E}\, M_n = \frac{n(n-1)}{4}, \tag{1.3} \qquad \boxed{\texttt{em}}$$

$$\operatorname{Var} M_n = \frac{n(n-1)(2n+5)}{72} = \frac{2n^3+3n^2-5n}{72}. \tag{1.4} \qquad \boxed{\texttt{varm}}$$

Even though there is no explicit expression for the coefficients themselves (i.e. for the exact probabilitity that a permutation of n objects would have a certain number of inversions), it is a classical result (see [3, Section X.6]), that follows from an extended form of the Central Limit Theorem, that the normalized version

$$\frac{M_n - n(n-1)/4}{\sqrt{(2n^3+3n^2-5n)/72}} \quad ,$$

tends to $N(0,1)$, as $n \to \infty$. So this sequence of probability distributions, too, is asymptotically normal.

But what about *words*, also known as *multi-set permutations*?. Permutations on n objects can be viewed as words in the alphabet $\{1, 2, \dots, n\}$, where each letter shows up *exactly* once. But what if we allow *repetitions*? I.e., we consider all words with a_1 occurrences of 1, a_2 occurrences of 2, …, a_m occurrences of m. (We assume throughout that $m \ge 2$ and each $a_j \ge 1$.) We all know that the number of such words is the multinomial coefficient

$$\binom{a_1 + \cdots + a_m}{a_1, \dots, a_m}$$

and many of us also know that the number of such words with exactly k inversions is the coefficient of q^k in the q-analog of the multinomial coefficient

$$\binom{a_1 + \cdots + a_m}{a_1, \dots, a_m}_q := \frac{[a_1 + \cdots + a_m]!}{[a_1]! \cdots [a_m]!} \quad , \tag{1.5} \qquad \boxed{\texttt{qmulti}}$$

where $[n]! := [1][2]\cdots[n]$, and $[n] := (1-q^n)/(1-q)$. Assuming that all words are equally likely (the uniform distribution), the probability generating function is

$$F_{a_1,\dots,a_m}(q) := \frac{\left(\prod_{i=1}^m a_i!\right) \cdot \prod_{i=1}^{a_1+\cdots+a_m}(1-q^i)}{(a_1+\cdots+a_m)! \prod_{j=1}^m \prod_{i=1}^{a_j}(1-q^i)} = \frac{F_{a_1+\cdots+a_m}(q)}{F_{a_1}(q)\cdots F_{a_m}(q)}. \tag{1.6} \qquad \boxed{\texttt{F}}$$

Indeed, this can be seen as follows. Let $M_{a_1,\dots,a_m}$ denote the number of inversions in a random word. If we distinguish the a_i occurrences of i by adding different fractional parts, in random order, the number of inversions will increase by Z_i, say, with the same distribution as M_{a_i}; further $M_{a_1,\dots,a_m}$ and $Z_1, \dots, Z_m$ are independent. On the other hand, $M_{a_1,\dots,a_m} + Z_1 + \cdots + Z_m$ has the same distribution as $M_{a_1+\cdots+a_m}$. Hence,

$$F_{a_1,\dots,a_m}(q) F_{a_1}(q) \cdots F_{a_m}(q) = F_{a_1+\cdots+a_m}(q), \tag{1.7} \qquad \boxed{\texttt{FF}}$$

which is (1.6).

By (1.6), we further have the factorization

$$F_{a_1,\ldots,a_m}(q) = \prod_{j=2}^{m} F_{A_{j-1},a_j}(q), \tag{1.8}$$ F2

where $A_j := a_1 + \cdots + a_j$, which reduces the general case to the two-letter case.

Note that (1.6) shows that the distribution of $M_{a_1,\ldots,a_m}$ is invariant if we permute $a_1, \ldots, a_m$; a symmetry which is not obvious from the definition.

Rpart **Remark 1.1.** The two-letter case is particularly interesting, since the unnormalized generating function

$$\binom{a+b}{a} F_{a,b}(q) = \frac{(1-q^{a+b})(1-q^{a+b-1})\cdots(1-q^{a+1})}{(1-q^{b})(1-q^{b-1})\cdots(1-q^{1})} = \frac{[a+b]!}{[a]!\,[b]!} \quad ,$$

(the q-binomial coefficient in (1.5)) is the same as the generating function for the set of integer-partitions with largest part $\leq a$ and $\leq b$ parts, in other words the set of integer-partitions whose Ferrers diagram lies inside an a by b rectangle, where the random variable is the "number of dots" (i.e. the integer being partitioned). In other words, the number of such partitions of an integer n equal the number of words of a 1's and b 2's with n inversions.

It is easy to see that the *mean* of $M_{a_1,\ldots,a_m}$ is

$$\mu(a_1,\ldots,a_m) := \mathbb{E}\, M_{a_1,\ldots,a_m} = e_2(a_1,\ldots,a_m)/2$$

(here $e_k(a_1,\ldots,a_m)$ is the degree k elementary symmetric function), so considering the shifted random variable $M_{a_1,\ldots,a_m} - \mu(a_1,\ldots,a_m)$, "number of inversions minus the mean", we get that the probability generating function is

$$G_{a_1,\ldots,a_m}(q) := q^{-\mu(a_1,\ldots,a_m)} F_{a_1,\ldots,a_m}(q) = \frac{F_{a_1,\ldots,a_m}(q)}{q^{e_2(a_1,\ldots,a_k)/2}} \tag{1.9}$$ G

By computing $(q(qG)')'$ and plugging-in $q = 1$, or from (1.7) and (1.3)–(1.4), it is easy to see that the *variance* $\sigma^2 := \operatorname{Var} M_{a_1,\ldots,a_m}$ is

$$\sigma^2 = \frac{(e_1+1)e_2 - e_3}{12} \quad . \tag{1.10}$$ sigma

(By σ we mean $\sigma(a_1,\ldots,a_m)$ and we omit the arguments $(a_1,\ldots,a_m)$ from the e_i's.)

Let $N := e_1 = a_1 + \cdots + a_m$, the length of the random word, and let $a^* := \max_j a_j$ and $N_* := N - a^*$.

One main result of the present article is:

T1 **Theorem 1.2.** *Consider the random variable, $M_{a_1,\ldots,a_m}$, "number of inversions", on the (uniform) sample space of words with a_1 1's, a_2 2's, …, a_m m's. For any sequence of sequences $(a_1,\ldots,a_m) = (a_1^{(\nu)},\ldots,a_{m^{(\nu)}}^{(\nu)})$ such* *that* $N_* := N - a^* \to \infty$*, the sequence of normalized random variables*

$$X_{a_1,\dots,a_m} = \frac{M_{a_1,\dots,a_m} - \mu(a_1,\dots,a_m)}{\sigma(a_1,\dots,a_m)} \quad ,$$

tends to the standard normal distribution $\mathcal{N}(0,1)$*, as* $\nu \to \infty$*.*

Theorem 1.2 includes both the case when $m \geq 2$ is fixed, and the case when $m \to \infty$. If m is fixed and $a_1 \geq a_2 \geq \cdots \geq a_m$, as may be assumed by symmetry, then the condition $N_* \to \infty$ is equivalent to $a_2 \to \infty$. In the case $m \to \infty$, the assumption $N_* \to \infty$ is redundant, because $N_* \geq m - 1$.

Remark 1.3. The condition $N_* \to \infty$ is also necessary for asymptotic normality, see Section 6.

We give a short proof of this result using characteristic functions in Section 3. We give first in Section 2 another proof (at least of a special case) that is *computer-assisted*, using the Maple package `MahonianStat` available from the webpage of this article:
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/mahon.html`, where one can also find sample input and output. This first proof uses the *method of moments.*

We conjecture that Theorem 1.2 can be refined to a local limit theorem as follows:

Conjlocal **Conjecture 1.4.** *Uniformly for all* $a_1,\dots,a_m$ *and all integers* k*,*

$$\mathbb{P}(M_{a_1,\dots,a_m} = k) = \frac{1}{\sqrt{2\pi}\sigma}\Big(e^{-(k-\mu)^2/(2\sigma^2)} + O\Big(\frac{1}{N_*}\Big)\Big). \tag{1.11}$$ local

We have not been able to prove this conjecture in full generality, but we prove it under additional hypotheses on $a_1,\dots,a_m$ in Section 4.

2. A computer-inspired proof

Spf1

We assume for simplicity that m is fixed, and that $(a_1,\dots,a_m) = (ta_1^0,\dots,ta_m^0)$ for some fixed $a_1^0,\dots,a_m^0$ and $t \to \infty$.

We discover and prove the leading term in the asymptotic expansion, in t, for an *arbitrary* $2r$-th moment, for the normalized random variable $X_{a_1,\dots,a_m} = (M_{a_1,\dots,a_m} - \mu)/\sigma$, and show that it converges to the moment $\mu_{2r} = (2r)!/(2^r r!)$ of $\mathcal{N}(0,1)$, for every r.

For the sake of exposition, we will only treat in detail the two-letter case, where we can find *explicit* expressions for the asymptotics of the $2r$-th moment, for symbolic a_1, a_2, t and r to *any* desired (specific) order s (i.e. the leading coefficient t^{3r} as well as the terms involving $t^{3r-1},\dots,t^{3r-s}$). A modified argument works for the general case, but we can only find the leading term, i.e. that

$$\alpha_{2r} := \mathbb{E}(X_{a_1,\dots,a_m})^{2r} = \frac{(2r)!}{2^r r!}t^{3r} + O(t^{3r-1}) \quad .$$

Of course the odd moments are all zero, since the distribution of $M_{a_1,\dots,a_m}$ is symmetric about μ.

The mean in the two-letter case is simply $ab/2$, so the probability generating function for $M_{a,b} - \mu$ is

$$G_{a,b}(q) = \frac{F_{a,b}(q)}{q^{ab/2}(a+b)!/(a!b!)} = \frac{a!b!(1-q^{a+b})(1-q^{a+b-1})\cdots(1-q^{a+1})}{q^{ab/2}(a+b)!(1-q^{b})(1-q^{b-1})\cdots(1-q^{1})} \quad ,$$

Taking ratios, we have:

$$\frac{G_{a,b}(q)}{G_{a-1,b}(q)} = \frac{a(1-q^{a+b})}{q^{b/2}(a+b)(1-q^{a})} \quad . \tag{2.1}$$ ratio

Recall that the binomial moments $A_r := \mathbb{E}[\binom{X}{r}]$ are the Taylor coefficients of the probability generating function (in our case $G_{a,b}(q)$) around $q = 1$. Writing $q = 1 + z$, we have

$$G_{a,b}(1+z) = \sum_{r=0}^{\infty} A_r(a,b) z^r$$

Note that $A_0(a,b) = 1$ and $A_1(a,b) = 0$. Let us call the expression on the right side of (2.1), with q replaced by $1+z$, $P(a,b,z)$:

$$P(a,b,z) := \frac{a(1-(1+z)^{a+b})}{(1+z)^{b/2}(a+b)(1-(1+z)^{a})}$$

Maple can easily expand $P(a,b,z)$ to any desired power of z, It starts out with

$$P(a,b,z) = 1 + \frac{1}{24}(2a+b)bz^2 - \frac{1}{24}(2a+b)bz^3 - \frac{1}{5760}\left(8a^3 - 8a^2b - 12ab^2 - 3b^3 - 440a - 220b\right)bz^4 + \ldots$$

note that the coefficients of all the powers of z are polynomials in (a,b).

So let us write

$$P(a,b,z) = \sum_{i=0}^{\infty} p_i(a,b) z^i \quad ,$$

where $p_i(a,b)$ are certain polynomials that Maple can compute for any i, no matter how big.

Looking at the recurrence

$$G_{a,b}(1+z) = P(a,b,z) G_{a-1,b}(1+z) \quad ,$$

and comparing coefficients of z^r on both sides, we get

$$A_r(a,b) - A_r(a-1,b) = \sum_{s=1}^{r} A_{r-s}(a-1,b) p_s(a,b) \quad . \tag{2.2}$$ recurrence

Assuming that we already know the polynomials $A_{r-1}(a,b), A_{r-2}(a,b), \ldots, A_0(a,b)$, the left side is a certain specific polynomial in a and b, that Maple can easily compute, and then $A_r(a,b)$ is simply the indefinite sum of that polynomial, that Maple can do just as easily. So (2.2) enables us to get *explicit* expressions for the binomial moments $A_r(a,b)$ for *any* (numeric) r.

But what about the general (symbolic) r? It is too much to hope for the full expression, **but** we can easily conjecture as many leading terms as we wish. We first conjecture, and then immediately prove by induction, that for $r \geq 1$

$$A_{2r}(a,b) = \frac{1}{r!}\left(\frac{ab(a+b)}{24}\right)^r + \text{lower order terms}$$

$$A_{2r+1}(a,b) = \frac{-1}{(r-1)!}\left(\frac{ab(a+b)}{24}\right)^r + \text{lower order terms} \quad ,$$

where we can conjecture (by fitting polynomials in (a, b) to the data obtained from the numerical r's) any (finite, specific) number of terms.

Once we have asymptotics, to any desired order, for the binomial moments, we can easily compute the moments $\mu_r(a, b)$ of $M_{a,b} - \mu$ themselves, for *any* desired specific r and asymptotically, to any desired order. We do that by using the expressions of the powers as linear combination of falling-factorials (or equivalently binomials) in terms of Stirling numbers of the second kind, $S(n, k)$. Note that for the asymptotic expressions to any desired order, we can still do it symbolically, since for any specific m, $S(n, n - m)$ is a polynomial in n (that Maple can easily compute, symbolically, as a polynomial in n). In particular, the variance is:

$$\sigma^2 = \mu_2(a,b) = \frac{ab(a+b+1)}{12} \quad ,$$

in accordance with (1.10). In general we have $\mu_{2r+1}(a, b) = 0$, of course, and the six leading terms of $\mu_{2r}(at, bt)$ can be found in the webpage of this article. From this, Maple finds that $\alpha_{2r}(at, bt) := \mu_{2r}(at, bt)/\mu_2(at, bt)^r$ are given asymptotically (for fixed a, b and $t \to \infty$) by:

$$\alpha_{2r}(at,bt) = \frac{(2r)!}{2^r r!} \cdot \left(1 - \frac{r(r-1)\left(b^2+ab+a^2\right)}{5ab\left(a+b\right)} \cdot \frac{1}{t}\right) + O(t^{-2}) \quad .$$

In particular, as $t \to \infty$, they converge to the famous moments of $\mathcal{N}(0, 1)$. QED.

2.1. **The general case.** To merely prove asymptotic normality, one does not need a computer, since we only need the leading terms. The above proof can be easily adapted to the general case $(a_1, \ldots, a_m) = (ta_1^0, \ldots, ta_m^0)$. One simply uses induction on m, the number of different letters.

2.2. **The Maple package MahonianStat.** The Maple package `MahonianStat`, accompanying this article, has lots of features, that the readers can explore at their leisure. Once downloaded into a directory, one goes into a maple session, and types `read MahonianStat`. To get a list of the main procedures, type: `ezra();`. To get help with a specific procedure, type `ezra(ProcedureName);`. Let us just mention some of the more important procedures.

`AsyAlphaW2tS(r,a,b,t,s)`: inputs symbols `r,a,b,t` and a positive integer s, and outputs the asymptotic expansion, to order s, for α_{2r} $(=\mu_{2r}/\mu_2^r)$

`ithMomWktE(r,e,t)` the r-th moment about the mean of the number of inversions of $a_1 t$ 1's, ..., $a_m t$ n's in terms of the elementary symmetric functions, in $a_1, \dots, a_m$. Here r is a specific (numeric) positive integer, but e and t are symbolic.

`AppxWk(L,x)`: Using the asymptotics implied by the asymptotic normality of the (normalized) random variable under consideration, finds an approximate value for the number of words with $L[1]$ 1's, $L[2]$ 2's, ..., $L[n]$ n's with exactly x inversions. For example, try: `AppxWk([100,100,100],15000);`

For the two-lettered case, one can get better approximations, by procedure `BetterAppxW2`, that uses improved limit-distributions, using more terms in the probability density function.

The webpage of this article has some sample input and output.

Spf2

3. A general proof of Theorem 1.2

We have an exact formula (1.10) for the variance σ^2 of $M_{a_1,\dots,a_m}$. We first show that σ 2 is always of the order $\Theta(N^2 N_*)$.

Lsigma

Lemma 3.1. *For any $a_1, \dots, a_m$,*

$$\frac{N^2 N_*}{36} \le \sigma^2 \le \frac{(N+1)NN_*}{12} \le \frac{N^2 N_*}{6}.$$

Proof. For the upper bounds we assume, by symmetry, that $a_1 \ge \dots \ge a_m$. Then $a^* = a_1$ and

$$e_2 = a_1 \sum_{j=2}^m a_j + a_2 \sum_{j=3}^m a_j + \dots \le N \sum_{j=2}^m a_j = NN_*.$$

Since $e_1 = N$, (1.10) yields the upper bounds.

For the lower bound, we first observe that $2e_2e_1 - 6e_3 \ge 0$ (since this difference can be written as a sum of certain $a_j a_k a_l$). Hence $e_3 \le e_1 e_2/3$ and (1.10) yields

$$12\sigma^2 \ge e_1 e_2 - e_3 \ge \tfrac{2}{3} e_1 e_2.$$

Further,

$$2e_2 = \sum_{j=1}^m a_j(N - a_j) \ge \sum_{j=1}^m a_j(N - a^*) = NN_*,$$

and the lower bound follows. □

Proof of Theorem 1.2. From (1.6) follows the identity

$$F_{n_1,n_2}(e^{\mathrm{i}\theta}) \ = \ \prod_{j=1}^{n_2} \frac{(e^{\mathrm{i}(n_1+j)\theta} - 1)/(\mathrm{i}(n_1+j)\theta)}{(e^{\mathrm{i}j\theta} - 1)/(\mathrm{i}j\theta)}. \qquad (3.1)$$ f2q

By Taylor's series

$$\log \frac{e^z - 1}{z} = z/2 + z^2/24 + O(z^4), \qquad |z| \le 1,$$

and we substitute this expansion into the identity (3.1) to conclude:

$$F_{n_1,n_2}(e^{\mathrm{i}\theta}) \;=\; \exp\left(\mathrm{i}n_1n_2\theta/2 - n_1n_2(n_1+n_2+1)\theta^2/24 + O(n_2n_1^4\theta^4)\right), \tag{3.2}$$

r3

uniformly for $n_1 \geq n_2 \geq 1$ and $|\theta| \leq (n_1+n_2)^{-1}$.

We use the factorization (1.8). By symmetry, we may assume $a_1 \geq a_2 \geq \cdots \geq a_m$, and then $A_{j-1} \geq a_{j-1} \geq a_j$ for each j. Thus (3.2) yields, uniformly for $q = e^{\mathrm{i}\theta}$ with $|\theta| \leq N^{-1}$,

$$\begin{aligned} F_{a_1,\dots,a_m}(q) &= \prod_{j=2}^{m} F_{A_{j-1},a_j}(q) \\ &= \exp\left(\sum_{j=2}^{m}\Big(\mathrm{i}A_{j-1}a_j\theta/2 - A_{j-1}a_j(A_j+1)\theta^2/24 + O(a_jA_{j-1}^4\theta^4)\Big)\right). \end{aligned}$$

Here, the sums of the coefficients of θ and θ^2 are easily evaluated, but we do not have to do that since they have to equal $\mathrm{i}\mu$ and $-\sigma^2/2$, respectively. Further,

$$\sum_{j=2}^{m} A_{j-1}^4 a_j \leq N^4 \sum_{j=2}^{m} a_j = N^4 N_*. \tag{3.3}$$

pa1

Consequently, if $|\theta| \leq N^{-1}$,

$$F_{a_1,\dots,a_m}(e^{\mathrm{i}\theta}) = \exp\big(\mathrm{i}\mu\theta - \sigma^2\theta^2/2 + O(N^4N_*\theta^4)\big) \tag{3.4}$$

pa2

and, by (1.9),

$$G_{a_1,\dots,a_m}(e^{\mathrm{i}\theta}) = \exp\big(-\sigma^2\theta^2/2 + O(N^4N_*\theta^4)\big). \tag{3.5}$$

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Let $\theta = t/\sigma$. For any fixed t, by Lemma 3.1,

$$|Nt/\sigma| = O\big({N_*}^{-1/2}\big) = o(1),$$

so $|\theta| \leq N^{-1}$ if ν is large enough. Hence, by (3.3)–(3.5) and Lemma 3.1,

$$G_{a_1,\dots,a_m}\big(e^{\mathrm{i}t/\sigma}\big) == \exp\left(-\frac{t^2}{2} + O\Big(\frac{N^4N_*t^4}{N^4{N_*}^2}\Big)\right) = \exp\big(-t^2/2 + o(1)\big),$$

and Theorem 1.2 follows by the continuity theorem [4, Theorem XV.3.2]. □

Slocal

4. The local limit theorem

"If one can prove a central limit theorem for a sequence $a_n(k)$ of numbers arising in enumeration, then one has a qualitative feel for their behavior. A local limit theorem is better because it provides asymptotic information about $a_n(k)$ …," [2]. In this section we prove that the relation (1.11) holds uniformly over certain very general, albeit not unrestricted, sets of tuples $\mathbf{a} = (a_1,\dots,a_m)$. The exact statement is given below in Theorem 4.5.

As explained in Bender [2], there are two standard conditions for passage from a central to a local limit theorem: (1) if the sequence in question is unimodal, then one has a local limit theorem for n in the set $\{|n-\mu| \geq \epsilon\sigma\}$, $\epsilon > 0$; (2) if the sequence in question is log-concave, then one has a local limit theorem for all n. Our sequence, the coefficients of the q-mutinomial, is in fact unimodal, as first shown by Schur [6] using invariant theory, and later by O'Hara [5] using combinatorics. Unfortunately, the ensuing local limit theorem fails to cover the most interesting coefficients, the largest ones, near the mean μ. However, our polynomials are manifestly not log-concave as is seen by inspecting the first three coefficients (assuming $n_1, n_2 \geq 2$)

$$\binom{n_1+n_2}{n_1}_q = 1 + q + 2q^2 + \cdots .$$

The question arises might the coefficients be log-concave near the mean, and here is a small table of empirical values: ($c[j] = [q^j]\binom{2n}{n}_q$)

n	$(c[n^2/2-1])^2 - c[n^2/2] \times c[n^2/2-2]$
2	-1
4	-7
6	-165
8	-1529
10	44160
12	7715737
14	905559058
16	101507214165
18	11955335854893
20	1501943866215277

Based on this scant evidence, we speculate that some sort of log-concavity theorem is true, but that its proper statement is complicated by describing the appropriate range of $\mathbf{a}$ and j. Thus, we use neither of the two standard methods mentioned above for proving our local limit theorem. (Later, we shall see that our theorem has implications for log-concavity.) Instead, we use another standard method, direct integration (Fourier inversion) of the characteristic function, or equivalently of the probability generating function $F(q)$ for $q = e^{\mathrm{i}\theta}$ on the unit circle. This method uses the estimates for small θ derived in Section 3 together with estimates of $|F(e^{\mathrm{i}\theta})|$ for larger θ. We begin with one such estimate for rather small θ.

LG1 **Lemma 4.1.** *There exists a constant $\tau > 0$ such that for any $a_1, \ldots, a_m$ and $|\theta| \leq \tau/N$,*

$$\left|F_{a_1,\ldots,a_m}(e^{\mathrm{i}\theta})\right| = \left|G_{a_1,\ldots,a_m}(e^{\mathrm{i}\theta})\right| \leq e^{-\sigma^2\theta^2/4}.$$

Proof. Suppose that $0 < |\theta| \leq \tau/N$. Then, using Lemma 3.1,

$$\frac{N^4 N_* \theta^4}{\sigma^2\theta^2} \leq \frac{N^2 N_* \tau^2}{\sigma^2} \leq 36\tau^2,$$

so if τ is chosen small enough, the error term $O(N^4 N_* \theta^4)$ in (3.4) and (3.5) is $\leq \sigma^2\theta^2/4$, and thus the result follows from (3.5). □

We let in the sequel τ denote this constant. We may assume $0 < \tau \le 1$.

Lmagnus **Lemma 4.2.** *Uniformly, for all $a_1, \dots, a_m$ and all integers k,*

$$\mathbb{P}(M_{a_1,\dots,a_m} = k) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(k-\mu)^2/(2\sigma^2)} + \int_{\tau/N}^{\pi} |F_{a_1,\dots,a_m}(e^{\mathrm{i}\theta})| \,\mathrm{d}\theta + O\Bigl(\frac{1}{\sigma N_*}\Bigr).$$

Proof. For any integer k,

$$\begin{aligned}
&\mathbb{P}(M_{a_1,\dots,a_m} = k) - \frac{1}{\sqrt{2\pi}\sigma} e^{-(k-\mu)^2/(2\sigma^2)} \\
&\quad= \frac{1}{2\pi}\int_{-\pi}^{\pi} F_{a_1,\dots,a_m}(e^{\mathrm{i}\theta}) e^{-\mathrm{i}k\theta}\,\mathrm{d}\theta - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\sigma^2\theta^2/2} e^{-\mathrm{i}(k-\mu)\theta}\,\mathrm{d}\theta \\
&\quad= \frac{1}{2\pi}\int_{|\theta|\le\tau/N} \Bigl(G_{a_1,\dots,a_m}(e^{\mathrm{i}\theta}) - e^{\sigma^2\theta^2/2}\Bigr) e^{-\mathrm{i}(k-\mu)\theta}\,\mathrm{d}\theta \\
&\qquad\quad + \frac{1}{2\pi}\int_{\tau/N\le|\theta|\le\pi} F_{a_1,\dots,a_m}(e^{\mathrm{i}\theta}) e^{-\mathrm{i}k\theta}\,\mathrm{d}\theta \\
&\qquad\quad - \frac{1}{2\pi}\int_{|\theta|\ge\tau/N} e^{-\sigma^2\theta^2/2} e^{-\mathrm{i}(k-\mu)\theta}\,\mathrm{d}\theta \\
&\quad=: I_1 + I_2 + I_3.
\end{aligned}$$

For $|\theta| \le \tau/N$, $|G_{a_1,\dots,a_m}(e^{\mathrm{i}\theta})| \le e^{-\sigma^2\theta^2/4}$ by Lemma 4.1, so (3.5) and the mean-value theorem applied to $z \mapsto e^z$ yield

$$\begin{aligned}
\Bigl|G_{a_1,\dots,a_m}(e^{\mathrm{i}\theta}) - e^{-\sigma^2\theta^2/2}\Bigr| &\le O(N^4 N_* \theta^4) \max(|G_{a_1,\dots,a_m}(e^{\mathrm{i}\theta})|, e^{-\sigma^2\theta^2/2}) \\
&= O\bigl(N^4 N_* \theta^4 e^{-\sigma^2\theta^2/4}\bigr).
\end{aligned}$$

Integrating, we find, using Lemma 4.1,

$$\begin{aligned}
|I_1| &\le \int_{|\theta|\le\tau/N} \Bigl|G_{a_1,\dots,a_m}(e^{\mathrm{i}\theta}) - e^{-\sigma^2\theta^2/2}\Bigr|\,\mathrm{d}\theta \\
&\le O\bigl(N^4 N_*\bigr) \int_{-\infty}^{\infty} \theta^4 e^{-\sigma^2\theta^2/4}\,\mathrm{d}\theta = O\bigl(N^4 N_* \sigma^{-5}\bigr) = O\Bigl(\frac{1}{\sigma N_*}\Bigr).
\end{aligned}$$

Further, again using Lemma 4.1,

$$|I_3| \le \int_{\tau/N}^{\infty} e^{-\sigma^2\theta^2/2}\,\mathrm{d}\theta \le 3\sigma^{-1} e^{-(\sigma\tau/N)^2/2} \le \frac{6}{\sigma(\sigma\tau/N)^2} = O\Bigl(\frac{1}{\sigma N_*}\Bigr).$$

Finally, $|I_2| \le \int_{\tau/N}^{\pi} |F_{a_1,\dots,a_m}(e^{\mathrm{i}\theta})|\,\mathrm{d}\theta$. □

In order to verify Conjecture 1.4, it thus suffices to show that the integral $\int_{\tau/N}^{\pi} |F_{a_1,\dots,a_m}(e^{\mathrm{i}\theta})|\,\mathrm{d}\theta$ in Lemma 4.2 is $O\bigl(\frac{1}{\sigma N_*}\bigr)$.

Remark 4.3. For example, an estimate

$$F_{a_1,\dots,a_m}(e^{\mathrm{i}\theta}) = O\Bigl(\frac{1}{\sigma^3\theta^3}\Bigr), \qquad 0 < \theta \le \pi, \tag{4.1}$$ conj3

is sufficient for (1.11). We conjecture that this estimate (4.1) holds when $N_* \ge 6$, say. Note that it does not hold for very small N_*: taking $\theta = \pi$ we have, for even n_1, $F_{n_1,1}(-1) = 1/(n_1+1) = 1/N$, and the same holds for $F_{n_1,2}(-1)$.

Note further that even the weaker estimate

$$F_{a_1,\dots,a_m}(e^{\mathrm{i}\theta}) = O\Big(\frac{1}{\sigma^2\theta^2}\Big), \qquad 0<\theta\le\pi, \tag{4.2}$$ conj2

would be enough to prove (1.11) with the weaker error term $O({N_*}^{-1/2})$.

We obtain a partial proof of Conjecture 1.4 using the following lemma.

LC1 **Lemma 4.4.** *For a given $\tau\in(0,1]$ there exists $c=c(\tau)>0$ such that*

$$|F_{n_1,n_2}(e^{\mathrm{i}\theta})| \;\le\; e^{-cn_2} \tag{4.3}$$ r2

for $n_1\ge n_2\ge 1$ and $\tau/(n_1+n_2)\le|\theta|\le\pi$.

More generally, for any $a_1,\dots,a_m$ and $\tau/N\le|\theta|\le\pi$,

$$|F_{a_1,\dots,a_m}(e^{\mathrm{i}\theta})| \;\le\; e^{-cN_*}. \tag{4.4}$$ r2m

Proof. We prove first (4.3). For positive integer n define

$$f_n(y,q) = \prod_{j=0}^{n}(1-yq^j)^{-1}.$$

For $0\le R<1$, we have (e.g. by Taylor expansions) $e^{2R}\le\frac{1+R}{1-R}$, and thus $e^{4R}\le\frac{(1+R)^2}{(1-R)^2} = 1+\frac{4R}{(1-R)^2}$. Hence, by convexity, for any real ζ,

$$e^{2R(1-\cos\zeta)} \le 1+\frac{2R(1-\cos\zeta)}{(1-R)^2} = \frac{1+R^2-2R\cos\zeta}{(1-R)^2} = \frac{|1-Re^{i\zeta}|^2}{(1-R)^2},$$

and thus

$$\Big|(1-Re^{\mathrm{i}\zeta})^{-1}\Big| \;\le\; (1-R)^{-1}\exp\left(R(1-\cos\zeta)\right).$$

Consequently, by a simple trigonometric identity, for any real ϕ and θ,

$$\begin{aligned}\Big|f_{n_1}(Re^{\mathrm{i}\phi},e^{\mathrm{i}\theta})\Big| \;&\le\; (1-R)^{-n_1-1}\\ &\times\exp\left(-R\Big(n_1+1-\cos\big(\phi+\frac{n_1}{2}\theta\big)\frac{\sin(n_1+1)\theta/2}{\sin\theta/2}\Big)\right)\\ &\le\; (1-R)^{-n_1-1}\times\exp\left(R\Big(-n_1-1+\frac{\sin(n_1+1)\theta/2}{\sin\theta/2}\Big)\right).\end{aligned}$$

The function $g(\theta)=g_n(\theta):=\frac{\sin n(\theta/2)}{\sin(\theta/2)}$, where $n\ge1$, is an even function of θ; is decreasing for $0\le\theta\le\pi/n$, as can be verified by calculating g'; and satisfies $|g(\theta)|\le g(\pi/n)$ for $\pi/n\le|\theta|\le\pi$. Further, for $n\ge2$ and $0\le|\theta|\le\pi/n$,

$$\begin{aligned}g_n(\theta) &= 2\frac{\sin(n\theta/4)}{\sin(\theta/2)}\cos(n\theta/4) = 2g_{n/2}(\theta)\cos(n\theta/4)\le n\cos(n\theta/4)\\ &\le n\Big(1-\frac{n^2\theta^2}{40}\Big).\end{aligned}$$

Let $\theta_0 = \tau(n_1+n_2)^{-1} < \pi/(n_1+1)$. For $\theta_0 \le |\theta| \le \pi$ we thus have

$$|g_{n_1+1}(\theta)| \le g_{n_1+1}(\theta_0) \ \le\ n_1+1-\frac{n_1^3\theta_0^2}{40};$$

whence, for $0 \le R < 1$, the estimate above yields

$$\Bigl|f_{n_1}(Re^{\mathrm{i}\phi}, e^{\mathrm{i}\theta})\Bigr| \ \le\ (1-R)^{-n_1-1}\,\exp\bigl(-Rn_1^3\theta_0^2/40\bigr). \tag{4.5}$$ tho

Combinatorially we know that $[y^\ell q^n]f_{n_1}(y,q)$ is the number of partitions of n having at most ℓ parts no one of which exceeds n_1. As said in Remark 1.1, this equals $[q^n]\binom{n_1+\ell}{n_1}F_{n_1+\ell}(q)$. Hence, using Cauchy's integral formula, for any $R>0$,

$$\binom{n_1+n_2}{n_1}F_{n_1,n_2}(q) = [y^{n_2}]f_{n_1}(y,q) = \frac{1}{2\pi\mathrm{i}}\int_{|y|=R} f_{n_1}(y,q)\frac{\mathrm{d}y}{y^{n_2+1}} \quad .$$

Consequently, (4.5) implies that for $\theta_0 \le |\theta| \le \pi$ and $0<R<1$,

$$\binom{n_1+n_2}{n_1}\bigl|F_{n_1,n_2}(q)\bigr| \le (1-R)^{-n_1-1}R^{-n_2}\,\exp\bigl(-Rn_1^3\theta_0^2/40\bigr).$$

Now choose $R = \rho := n_2/(n_1+n_2) \le 1/2$. By Stirling's formula,

$$\binom{n_1+n_2}{n_1} = \Omega\bigl(n_2^{-1/2}\bigr)\,(1-\rho)^{-n_1-1}\,\rho^{-n_2}$$

and thus, for $\theta_0 \le |\theta| \le \pi$,

$$\bigl|F_{n_1,n_2}(q)\bigr| \le O(n_2^{1/2})\,\exp\bigl(-\rho n_1^3\theta_0^2/40\bigr) = O(n_2^{1/2})\,\exp\left(-\Omega(n_2)\right).$$

This shows (4.3) for n_2 large enough, say $n_2 \ge n_0$. The case $n_2 \le n_0$ can be XXX

XXX
I hope this case is simple, and perhaps can be dismissed without giving details. If not, I think that we do not really need it.

To prove (4.4), we assume as we may that $a_1 \ge \cdots \ge a_m$ and use the factorization (1.8). Let J be the first index such that $a_2+\cdots+a_J \ge N_*/2$. For $j \ge J$, then $A_{j-1}+a_j = A_j \ge A_J \ge a_1 + N_*/2 \ge N/2$, and thus $A_k|\theta| \ge N|\theta|/2 \ge \tau/2$; hence (4.3) yields

$$\bigl|F_{A_{j-1},a_j}(e^{\mathrm{i}\theta})\bigr| \le e^{-c(\tau/2)a_j}.$$

We thus obtain from (1.8), since each F_{n_1,n_2} is a probability generating function and thus is bounded by 1 on the unit circle,

$$|F_{a_1,\dots,a_m}(e^{\mathrm{i}\theta})| = \prod_{j=2}^m |F_{A_{j-1},a_j}(e^{\mathrm{i}\theta})| \le \prod_{j=J}^m e^{-c(\tau/2)a_j} \le e^{-c(\tau/2)N_*/2},$$

because $\sum_{j=J}^m a_j \ge N_*/2$. This proves (4.4) (redefining $c(\tau)$). □

Tlocal **Theorem 4.5.** *There exists a positive constant c such that for every C, the following is true. Uniformly for all $a_1,\dots,a_m$ such that $a^* \le Ce^{cN_*}$ and all integers k,*

$$\mathbb{P}(M_{a_1,\dots,a_m}=k) = \frac{1}{\sqrt{2\pi}\sigma}\Bigl(e^{-(k-\mu)^2/(2\sigma^2)} + O\Bigl(\frac{1}{N_*}\Bigr)\Bigr). \tag{4.6}$$ localxxx

Proof. Let $c_1 = c(\tau)$ be the constant in Lemma 4.4. Then, Lemmas 4.2 and 4.4 yield

$$\mathbb{P}(M_{a_1,\dots,a_m} = k) = \frac{1}{\sqrt{2\pi}\sigma}\Big(e^{-(k-\mu)^2/(2\sigma^2)} + O\Big(\frac{1}{N_*} + \sigma e^{-c_1 N_*}\Big)\Big).$$

For any fixed $c < c_1$ we have, using Lemma 3.1, $\sigma N_* e^{-c_1 N_*} = O(N e^{-cN_*})$ and thus

$$\mathbb{P}(M_{a_1,\dots,a_m} = k) = \frac{1}{\sqrt{2\pi}\sigma}\Big(e^{-(k-\mu)^2/(2\sigma^2)} + O\Big(\frac{1 + Ne^{-cN_*}}{N_*}\Big)\Big).$$

The result follows, since $Ne^{-cN_*} = a^* e^{-cN_*} + N_* e^{-cN_*} = a^* e^{-cN_*} + O(1)$. □

`[Old stuff to be edited:]` XXX

The combinatorics of the Macmahon q statistic appear not to have been greatly used, but in fact play an essential, easily overlooked, role. It seems that the Taylor estimate (3.2) is the key for a central limit theorem. As in Bender [2], some kind of smoothness condition is needed to proceed to a local limit theorem (1). Lemma 1, the large θ bound, imposes the required smoothness. The proof of Lemma 1 is made possible by the following combinatorial property:

$$[q^n]\binom{n_1+n_2}{n_1}_q = \sum_{\ell=1}^{n_2}[y^\ell]\prod_{j=1}^{n_1}(1-yq^j)^{-1}.$$

Thus, our Theorem 4.5 may be the $h(x) = (1-x)^{-1}$ case of a more general theorem about

$$[q^n]\sum_{\ell=1}^{n_2}[y^\ell]\prod_{j=1}^{n_1}h(yq^j).$$

5. Log-concavity

`[Old stuff to be edited:]` XXX

Let us review the proof of Theorem 4.5 with the intention of greater accuracy. The goal is to prove log-concavity in some range. For concreteness, let $\mathbf{a} = (n,n)$. Then σ^2 is of order n^3, and for sufficient accuracy we take the Taylor series (3) out to $O(\theta^{10})$. This improves (5) to:

$$\frac{1}{2\pi}\int_{\text{small}\,\theta} F(e^{\mathrm{i}\theta})e^{-\mathrm{i}j\theta}d\theta \;=\; \frac{F(1)}{\sigma\sqrt{2\pi}}\left(e^{-x^2/2}\left(1+\frac{p_4(x)}{\sigma^4}+\frac{p_6(x)}{\sigma^6}+\frac{p_8(x)}{\sigma^8}\right)+O(\frac{1}{n^4})\right).$$

Here, $p_4(x), \dots$ are certain polynomials and x is determined by $j = \mu + x\sigma$. The sizes of the terms involving the polynomials are $1, 1/n, 1/n^2, 1/n^3$. The large θ contribution is negligible by comparison. One may now use the formula to calculate the difference $c_j^2 - c_{j_1}c_{j+1}$ to accuracy $O(1/n^4)$. Taking out the obvious common factors, it is found to equal $1/\sigma^2$. This gives:

TC2 **Theorem 5.1** (A log-concavity result)**.** *For each constant C we have n_0 such that for $n \ge n_0$*

$$c_j^2 \ge c_{j-1}c_{j+1},$$

for $|j-\mu| \le C\sigma$. Here,

$$c_j = [q^j]\binom{2n}{n}_q.$$

We note that the "mysterious" numbers appearing in our earlier table are asymptotically

$$\frac{1}{2\pi\sigma^4}\binom{2n}{n}^2.$$

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6. Final comments

Suppose that $N_* \not\to \infty$. We may, as usual, assume that $a_1 \geq \cdots \geq a_m$. By considering a subsequence (if necessary), we may assume that $N_* := N - a^* = a_2 + \cdots + a_m$ is a constant; this entails that m is bounded, so by again considering a subsequence, we may assume that m and $a_2, \ldots, a_m$ are constant. We thus study the case when $a_1 \to \infty$ with fixed $a_2, \ldots, a_m$.

In this case, the number of inversions between indices $2, \ldots, m$ is $O(1)$, which is asymptotically negligible. Ignoring these, we can thus consider the random word as N_* letters $2, \ldots, m$ inserted in a_1 1's, and the number of inversions is the sum of their positions, counted from the end. It follows easily, either probabilistically or by calculating the characteristic function from (1.6), that $M_{a_1,\ldots,a_m}/N$, or equivalently $M_{a_1,\ldots,a_m}/a_1$, converges in distribution to the sum $\sum_{j=1}^{N_*} U_j$ of N_* independent random variables U_j with the uniform distribution on $[0,1]$. Equivalently, since $\sigma^2 \sim n_1^2 N_*/12 \sim N^2 N_*/12$,

$$\frac{M_{a_1,\ldots,a_m} - \mu(a_1,\ldots,a_m)}{\sigma(a_1,\ldots,a_m)} \xrightarrow{\mathrm{d}} \sqrt{\frac{12}{N_*}} \sum_{j=1}^{N_*} (U_j - \tfrac{1}{2}),$$

where $\xrightarrow{\mathrm{d}}$ denotes convergence in distribution. This limit is clearly not normal for any finite N_*. (However, its distribution is close to standard normal for large N_*. Note that it is normalized to mean 0 and variance 1.)

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