Optimizing a losing game

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1 The problem

Stan Wagon told me about the following pretty coin tossing problem. Suppose Alice has a coin with heads probability q and Bob has one with heads probability p. Suppose $q < p$. Now each of them will toss their coin n times, and Alice wins iff she gets more heads than Bob does. Evidently the game favors Bob, but for the given p, q, what is the choice of n that maximizes Alice's chances of winning?

Her chances of winning are

$$f(n) = \sum_{r \geq 0} \binom{n}{r} p^r (1-p)^{n-r} \sum_{s>r} \binom{n}{s} q^s (1-q)^{n-s}. \tag{1}$$

Our task will be to find a recurrence for $f(n)$.

2 The recurrence for the summand

Put

$$x = p/(1-p), \quad y = q/(1-q), \quad g(n) = f(n)/((1-p)^n (1-q)^n), \tag{2}$$

so

$$g(n) = \sum_{r \geq 0} \sum_{s>r} \binom{n}{r} \binom{n}{s} x^r y^s.$$

Let $G(n, r, s) = \binom{n}{r}\binom{n}{s} x^r y^s$, be the summand. We use Zeilberger's algorithm, and his program `MulZeil` returns a recurrence

$$\begin{aligned} G(n+1, r, s) - (x+1)(y+1)G(n, r, s) &= (K_r - 1)(c_1(n, r, s)G(n, r, s)) \\ &\quad + (K_s - 1)(c_2(n, r, s)G(n, r, s)), \end{aligned} \tag{3}$$

where K_r, K_s are shift operators in their subscripts, and the c_i are given by

$$c_1 = c_1(n,r,s) = \frac{r(1+y)}{r-n-1}; \qquad c_2 = c_2(n,r,s) = \frac{s(n+1)}{(s-n-1)(n-r+1)}. \tag{4}$$

This is the recurrence for the summand, and it can be quickly verified by dividing through by $G(n,r,s)$, canceling all of the factorials, and noting that the resulting polynomial identity states that $0=0$.

3 The recurrence for the sum

To find the recurrence for the sum, first in the case where ties do not count for Bob, we sum the recurrence (3) over $s > r$, and then sum the result over $r \geq 0$. To do this we have first, for every function ϕ of compact support,

$$\sum_{r\geq 0}\sum_{s>r}(K_r - 1)\phi(r,s) = -\sum_{s\geq 1}\phi(0,s) + \sum_{r\geq 1}\phi(r,r),$$

and

$$\sum_{r\geq 0}\sum_{s>r}(K_s - 1)\phi(r,s) = -\sum_{r\geq 0}\phi(r,r+1).$$

Consequently there results

$$\begin{aligned} g(n+1) - (x+1)(y+1)g(n) \quad &= \quad -\sum_{s\geq 1} c_1(n,0,s)G(n,0,s) + \sum_{r\geq 1} c_1(n,r,r)G(n,r,r) \\ &\qquad -\sum_{r\geq 0} c_2(n,r,r+1)G(n,r,r+1). \end{aligned}$$

Next we insert the values, from (4),

$$c_1(n,0,s) = 0; \quad c_1(n,r,r) = \frac{r(1+y)}{r-n-1}; \qquad c_2(n,r,r+1) = \frac{(r+1)(n+1)}{(r-n)(n-r+1)},$$

which gives

$$g(n+1) - (x+1)(y+1)g(n) = \sum_{r\geq 1}\frac{r(1+y)}{r-n-1}G(n,r,r) - \sum_{r\geq 0}\frac{(r+1)(n+1)}{(r-n)(n-r+1)}G(n,r,r+1).$$

Now substitute the values $G(n,r,r) = \binom{n}{r}^2 (xy)^r$, and $G(n,r,r+1) = \binom{n}{r}\binom{n}{r+1} x^r y^{r+1}$, and simplify the result, to obtain

$$\begin{aligned} g(n+1) - (x+1)(y+1)g(n) &= \sum_{r\geq 1} \frac{r(1+y)}{r-n-1} \binom{n}{r}^2 (xy)^r \\ &\quad - \sum_{r\geq 0} \frac{(r+1)(n+1)}{(r-n)(n-r+1)} \binom{n}{r}\binom{n}{r+1} x^r y^{r+1} \\ &= -(y+1)\sum_{r\geq 0} \binom{n}{r+1}\binom{n}{r} x^{r+1} y^{r+1} + \sum_{r\geq 0} \binom{n+1}{r}\binom{n}{r} x^r y^{r+1} \\ &= y\phi_n(xy) - \psi_n(xy), \end{aligned}$$

say, where

$$\phi_n(z) = \sum_{r=0}^{n} \binom{n}{r}^2 z^r; \qquad \psi_n(z) = \sum_{r=0}^{n} \binom{n}{r+1}\binom{n}{r} z^{r+1}. \tag{5}$$

Next, replace $g(n)$ by $f(n)/((1-p)^n(1-q)^n)$, noting that $(x+1)(y+1) = 1/((1-p)(1-q))$, to get

$$\frac{f(n+1) - f(n)}{((1-p)(1-q))^{n+1}} = y\phi_n(xy) - \psi_n(xy) = \frac{q}{1-q}\phi_n\left(\frac{pq}{(1-p)(1-q)}\right) - \psi_n\left(\frac{pq}{(1-p)(1-q)}\right). \tag{6}$$

Note that this gives a rapid method of computing $f(n)$ since each of the polynomials ϕ, ψ satisfies a three term recurrence, namely

$$(n+2)\phi_{n+2}(z) = (z+1)(2n+3)\phi_{n+1}(z) - (z-1)^2(n+1)\phi_n(z),$$

and

$$(n+1)(n+3)\psi_{n+2}(z) = (n+2)(2n+3)(z+1)\psi_{n+1}(z) - (z-1)^2(n+2)(n+1)\psi_n(z),$$

with initial conditions

$$\phi_0(z) = 1;\ \phi_1(z) = 1+z;\ \psi_0(z) = 0;\ \psi_1(z) = z.$$

4 Analysis of the result

Regard p, q as fixed. Since $f(1) - f(0) > 0$, f is initially increasing. We want to discover the circumstances under which f will be decreasing steadily when n is large enough. This means that we want to analyze $y\phi_n(z) - \psi_n(z)$ for fixed z and large n. To do this we show first that both the ϕ's and the ψ's are closely related to the Legendre polynomials.

First, it is well known that

$$\phi_n(z) = (1-z)^n P_n\left(\frac{1+z}{1-z}\right).$$

Next, we have $\psi_n'(z) = n\phi_n(z) - z\phi_n'(z)$, from which

$$\begin{aligned}
\psi_n(z) &= (n+1)\int_0^z \phi_n(t)dt - z\phi_n(z) \\
&= (n+1)\int_0^z (1-t)^n P_n\left(\frac{1+t}{1-t}\right) dt - z\phi_n(z) \\
&= (n+1)2^{n+1}\int_1^{\frac{1+z}{1-z}} \frac{P_n(u)du}{(1+u)^{n+2}} - z(1-z)^n P_n\left(\frac{1+z}{1-z}\right),
\end{aligned}$$

thereby expressing the ψ's in terms of the Legendre polynomials also.

5 Hypergeometric functions

Directly from its definition (5) we have that $\psi_n(z) = nz\,{}_2F_1[-n, 1-n; 2|z]$. On the other hand, from Rainville we have

$${}_2F_1\left[-n, 1-n; 2\middle|\frac{1+x}{x-1}\right] = \frac{1}{n+1}\left(\frac{2}{1-x}\right)^n P_n^{(1,-1)}(x),$$

in which the P's on the right are the Jacobi polynomials.