Optimizing a losing game

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1 The problem

Stan Wagon told me about the following pretty coin tossing problem. Suppose Alice has a
coin with heads probability \( q \) and Bob has one with heads probability \( p \). Suppose \( q < p \).
Now each of them will toss their coin \( n \) times, and Alice wins iff she gets more heads than
Bob does. Evidently the game favors Bob, but for the given \( p, q \), what is the choice of \( n \) that
maximizes Alice’s chances of winning?

Her chances of winning are

\[
f(n) = \sum_{r \geq 0} \binom{n}{r} p^r (1 - p)^{n-r} \sum_{s > r} \binom{n}{s} q^s (1 - q)^{n-s}.
\]

(1)

Our task will be to find a recurrence for \( f(n) \).

2 The recurrence for the summand

Put

\[
x = p/(1 - p), \quad y = q/(1 - q), \quad g(n) = f(n)/(p^n (1 - q)^n),
\]

(2)

so

\[
g(n) = \sum_{r \geq 0} \sum_{s > r} \binom{n}{r} \binom{n}{s} x^r y^s.
\]

Let \( G(n, r, s) = \binom{n}{r} \binom{n}{s} x^r y^s \), be the summand. We use Zeilberger’s algorithm, and his
program \texttt{MulZeil} returns a recurrence

\[
G(n+1, r, s) - (x+1)(y+1)G(n, r, s) = (K_r - 1)(c_1(n, r, s)G(n, r, s))
+ (K_s - 1)(c_2(n, r, s)G(n, r, s)),
\]

(3)
where $K_r, K_s$ are shift operators in their subscripts, and the $c_i$ are given by
\begin{align*}
c_1 &= c_1(n, r, s) = \frac{r(1+y)}{r - n - 1}; \quad c_2 = c_2(n, r, s) = \frac{s(n + 1)}{(s - n - 1)(n - r + 1)}, \quad (4)
\end{align*}
This is the recurrence for the summand, and it can be quickly verified by dividing through by $G(n, r, s)$, canceling all of the factorials, and noting that the resulting polynomial identity states that $0 = 0$.

## 3 The recurrence for the sum

To find the recurrence for the sum, first in the case where ties do not count for Bob, we sum the recurrence (3) over $s > r$, and then sum the result over $r \geq 0$. To do this we have first, for every function $\phi$ of compact support,
\begin{align*}
\sum_{r \geq 0} \sum_{s > r} (K_r - 1) \phi(r, s) &= - \sum_{s \geq 1} \phi(0, s) + \sum_{r \geq 1} \phi(r, r),
\end{align*}
and
\begin{align*}
\sum_{r \geq 0} \sum_{s > r} (K_s - 1) \phi(r, s) &= - \sum_{r \geq 0} \phi(r, r + 1).
\end{align*}
Consequently there results
\begin{align*}
g(n + 1) - (x + 1)(y + 1)g(n) &= - \sum_{s \geq 1} c_1(n, 0, s) G(n, 0, s) + \sum_{r \geq 1} c_1(n, r, r) G(n, r, r) \\
&\quad - \sum_{r \geq 0} c_2(n, r, r + 1) G(n, r, r + 1).
\end{align*}
Next we insert the values, from (4),
\begin{align*}
c_1(n, 0, s) &= 0; \quad c_1(n, r, r) = \frac{r(1+y)}{r - n - 1}; \quad c_2(n, r, r + 1) = \frac{(r+1)(n+1)}{(r-n)(n-r+1)},
\end{align*}
which gives
\begin{align*}
g(n + 1) - (x + 1)(y + 1)g(n) &= \sum_{r \geq 1} \frac{r(1+y)}{r - n - 1} G(n, r, r) - \sum_{r \geq 0} \frac{(r+1)(n+1)}{(r-n)(n-r+1)} G(n, r, r + 1).
\end{align*}
Now substitute the values $G(n, r, r) = \binom{n}{r}^2 (xy)^r$, and $G(n, r, r + 1) = \binom{n}{r} \binom{n}{r+1} x^r y^{r+1}$, and simplify the result, to obtain

$$g(n + 1) - (x + 1)(y + 1)g(n) = \sum_{r \geq 1} \frac{r(1+y)}{r-n-1} \binom{n}{r}^2 (xy)^r - \sum_{r \geq 0} \frac{(r+1)(n+1)}{(r-n)(n-r+1)} \binom{n}{r} x^r y^{r+1}$$

$$= -(y + 1) \sum_{r \geq 0} \binom{n}{r+1} x^r y^{r+1} + \sum_{r \geq 0} \binom{n+1}{r} \binom{n}{r} x^r y^{r+1}$$

$$= y \phi_n(xy) - \psi_n(xy),$$

say, where

$$\phi_n(z) = \sum_{r=0}^{n} \binom{n}{r}^2 z^r; \quad \psi_n(z) = \sum_{r=0}^{n} \binom{n}{r+1} \binom{n}{r} z^{r+1}. \quad (5)$$

Next, replace $g(n)$ by $f(n)/((1-p)^n(1-q^n))$, noting that $(x+1)(y+1) = 1/((1-p)(1-q))$, to get

$$\frac{f(n+1) - f(n)}{(1-p)(1-q))^{n+1}} = y \phi_n(xy) - \psi_n(xy) = \frac{q}{1-q} \phi_n \left( \frac{pq}{(1-p)(1-q)} \right) - \psi_n \left( \frac{pq}{(1-p)(1-q)} \right). \quad (6)$$

Note that this gives a rapid method of computing $f(n)$ since each of the polynomials $\phi, \psi$ satisfies a three term recurrence, namely

$$(n + 2)\phi_{n+2}(z) = (z + 1)(2n + 3)\phi_{n+1}(z) - (z - 1)^2(n + 1)\phi_n(z),$$

and

$$(n + 1)(n + 3)\psi_{n+2}(z) = (n + 2)(2n + 3)(z + 1)\psi_{n+1}(z) - (z - 1)^2(n + 2)(n + 1)\psi_n(z),$$

with initial conditions

$$\phi_0(z) = 1; \quad \phi_1(z) = 1 + z; \quad \psi_0(z) = 0; \quad \psi_1(z) = z.$$

### 4 Analysis of the result

Regard $p, q$ as fixed. Since $f(1) - f(0) > 0$, $f$ is initially increasing. We want to discover the circumstances under which $f$ will be decreasing steadily when $n$ is large enough. This
means that we want to analyze \( y\phi_n(z) - \psi_n(z) \) for fixed \( z \) and large \( n \). To do this we show first that both the \( \phi \)'s and the \( \psi \)'s are closely related to the Legendre polynomials.

First, it is well known that

\[
\phi_n(z) = (1 - z)^n P_n \left( \frac{1 + z}{1 - z} \right).
\]

Next, we have \( \psi'_{n}(z) = n\phi_{n}(z) - z\phi'_{n}(z) \), from which

\[
\psi_{n}(z) = (n + 1) \int_{0}^{z} \phi_{n}(t) dt - z\phi_{n}(z)
\]

\[
= (n + 1) \int_{0}^{z} (1 - t)^n P_n \left( \frac{1 + t}{1 - t} \right) dt - z\phi_{n}(z)
\]

\[
= (n + 1) 2^{n+1} \int_{1}^{1+z} \frac{P_n(u) du}{(1 + u)^{n+2}} - z(1 - z)^n P_n \left( \frac{1 + z}{1 - z} \right),
\]

thereby expressing the \( \psi \)'s in terms of the Legendre polynomials also.

## 5 Hypergeometric functions

Directly from its definition (5) we have that \( \psi_{n}(z) = nz_{2} F_{1}[-n, 1 - n; 2|z] \). On the other hand, from Rainville we have

\[
_{2} F_{1} \left[ -n, 1 - n; 2 \frac{1 + x}{|x - 1|} \right] = \frac{1}{n + 1} \left( \frac{2}{1 - x} \right)^{n} P^{(1,-1)}_{n}(x),
\]

in which the \( P \)'s on the right are the Jacobi polynomials.