The Discrete Analog of the Malgrange-Ehrenpreis Theorem

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In fond memory of Leon Ehrenpreis

Abstract One of the landmarks of the modern theory of partial differential equations is the Malgrange-Ehrenpreis theorem that states that every non-zero linear partial differential operator with constant coefficients has a Green function (alias fundamental solution). In this short note I state the discrete analog, and give two proofs. The first one is Ehrenpreis- style, using duality, and the second one is constructive, using formal Laurent series.

Key Words Formal Laurent Series • systems of constant-coefficient partial differential equations • fundamental solution

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One of the **landmarks** of the modern theory of partial **differential** equations is the **Malgrange-Ehrenpreis**[E1][E2][M] theorem (see [Wi]) that states that every non-zero linear partial differential operator with constant coefficients has a Green's function (alias **fundamental solution**). Recently Wagner[W] gave an elegant *constructive* proof.

In this short note I will state the **discrete** analog, and give two proofs. The first one is Ehrenpreis-style, using *duality*, and the second one is *constructive*, using *formal Laurent series*.

Let Z be the set of integers, and n a positive integer. Consider functions $f(m_1, \ldots, m_n)$ from Z^n to the complex numbers (or any field). A **linear partial difference operator with constant coefficients** $\mathcal{P}$ is anything of the form

$$\mathcal{P}f(m_1, \ldots, m_n) := \sum_{\alpha \in A} c_\alpha f(m_1 + \alpha_1, \ldots, m_n + \alpha_n) \quad ,$$

where A is a **finite** subset of Z^n and $\alpha = (\alpha_1, \ldots, \alpha_n)$, and the c_α are **constants**.

For example, the **discrete Laplace operator** in two dimensions:

$$f(m_1, m_2) \to f(m_1, m_2) - \frac{1}{4}(f(m_1 + 1, m_2) + f(m_1 - 1, m_2) + f(m_1, m_2 + 1) + f(m_1, m_2 - 1)) \quad .$$

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The **symbol** of the operator $\mathcal{P}$ is the **Laurent polynomial**

$$P(z_1, \ldots, z_n) = \sum_{\alpha \in A} c_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n} \quad .$$

The **discrete delta function** is defined in the obvious way

$$\delta(m_1, \ldots, m_n) = \begin{cases} 1, & \text{if } (m_1, \ldots, m_n) = (0, 0, \ldots, 0); \\ 0, & \text{otherwise.} \end{cases}$$

Note that the beauty of the discrete world is that the delta function is a *genuine* function, not a "generalized" one, and one does not need the intimidating edifice of Schwartzian *distributions*.

We are now ready to state the

Discrete Malgrange-Ehrenpreis Theorem: Let $\mathcal{P}$ be any non-zero **linear partial difference operator with constant coefficients**. There exists a function $f(m_1, \ldots, m_n)$ defined on Z^n such that

$$\mathcal{P} f(m_1, \ldots, m_n) = \delta(m_1, \ldots, m_n) \quad .$$

First Proof (Ehrenpreis-style) Consider the infinite-dimensional vector space, $C[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}]$, of *all* Laurent polynomials in $z_1, \ldots, z_n$. Every function f on Z^n uniquely defines a **linear functional** T_f defined on monomials by

$$T_f \, [\, z_1^{m_1} \cdots z_n^{m_n} \,] := f(m_1, \ldots, m_n) \quad ,$$

and extended by linearity. Conversely, any linear functional gives rise to a function on Z^n. Let $P(z_1, \ldots, z_n)$ be the symbol of the operator $\mathcal{P}$. We are looking for a linear functional T such that for every monomial $z_1^{m_1} \cdots z_n^{m_n}$

$$T \, [\, P(z_1, \ldots, z_n) z_1^{m_1} \cdots z_n^{m_n} \,] = T_\delta(z_1^{m_1} \cdots z_n^{m_n}) \quad .$$

By linearity, for *any* Laurent polynomial $a(z_1, \ldots, z_n)$

$$T \, [\, P(z_1, \ldots, z_n) a(z_1, \ldots, z_n) \,] = T_\delta(a(z_1, \ldots, z_n)) \quad .$$

So T is defined on the (vector) **subspace** $P(z_1, \ldots, z_n) C[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}]$ of $C[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}]$. By elementary linear algebra, every linear functional on the former can be **extended** (in many ways!) to the latter. **QED**.

Before embarking on the second proof we have to recall the notion of **formal power series**, and more generally, **formal Laurent series**.

A *formal power series* in one variable z is any creature of the form

$$\sum_{i=0}^{\infty} a_i z^i \quad .$$

More generally, a **positive** *formal Laurent series* is any creature of the form

$$\sum_{i=m}^{\infty} a_i z^i \quad ,$$

where m is a (possibly negative) integer. On the other hand a **negative** *formal Laurent series* is any creature of the form

$$\sum_{i=-\infty}^{m} a_i z^i \quad ,$$

where m is a (possibly positive) integer.

A **bilateral** formal Laurent series goes **both ways**

$$\sum_{i=-\infty}^{\infty} a_i z^i \quad .$$

Note that the class of bilateral formal Laurent series is an abelian additive group, but one **can't** multiply there. On the other hand one can legally multiply two positive formal Laurent series by each other, and two negative formal Laurent series by each other, but don't mix them! Of course it is always legal to multiply any Laurent polynomial by any bilateral formal power series. **But watch out for** *zero-divisors*, e.g.

$$(1-z) \sum_{i=-\infty}^{\infty} z^i \, = \, 0 \quad .$$

Any Laurent polynomial $p(z) = a_i z^i + \dots a_j z^j$ of low-degree i and (high-)degree j in z (so $a_i \neq 0$, $a_j \neq 0$) has **two** natural multiplicative inverses. One in the ring of positive Laurent polynomials, and the other in the ring of negative Laurent polynomials. Simply write $p(z) = z^i a_i p_0(z)$ and get $1/p(z) = z^{-i}(1/a_i) p_0(z)^{-1}$, and writing $p_0(z) = 1 + q_0(z)$, we form

$$p_0(z)^{-1} = (1 + q_0(z))^{-1} = \sum_{i=0}^{\infty} (-1)^i q_0(z)^i \quad ,$$

and this makes perfect sense and *converges* in the ring of formal power series. Analogously one can form a multiplicative inverse in powers in z^{-1}.

It follows that every *rational function* $P(z)/Q(z)$ in one variable, z, has two natural inverses, one pointing positively, one negatively.

What about a rational function of several variables, $P(z_1, \dots, z_n)/Q(z_1, \dots, z_n)$? Here we can form $2^n n!$ natural inverses. There are $n!$ ways to order the variables, and for each of these one can decide whether to do the positive-pointing inverse or the negative-pointing one. At each stage we get a one-sided formal Laurent series whose coefficients are rational functions of the remaining variables, and one just keeps going.

Second Proof (Constructive): To every discrete function $f(m_1,\dots,m_n)$ associate the bilateral formal Laurent series

$$\sum_{(m_1,\dots,m_n)\in Z^n} f(m_1,\dots,m_n) z_1^{m_1}\cdots z_n^{m_n} \quad .$$

We need to "solve" the equation

$$P(z_1^{-1},\dots,z_n^{-1})\left(\sum_{(m_1,\dots,m_n)\in Z^n} f(m_1,\dots,m_n) z_1^{m_1}\cdots z_n^{m_n}\right) = 1 \quad .$$

So "explicitly"

$$\sum_{(m_1,\dots,m_n)\in Z^n} f(m_1,\dots,m_n) z_1^{m_1}\cdots z_n^{m_n} = 1/P(z_1^{-1},\dots,z_n^{-1}) \quad ,$$

and we just described how to do it in $2^n n!$ ways.

The Maple package LEON

This article is accompanied by a Maple package `LEON`. One of its numerous procedures is `FS`, that implements the above constructive proof. `LEON` can also compute **polynomial bases** to solutions of linear partial difference equations with constant coefficients, compute *Hilbert Series* for spaces of solutions of systems of linear differential equations, as well as find "multiplicity varieties" (in the style of Ehrenpreis) when they are zero-dimensional.

Leon Ehrenpreis (1930-2010): a truly FUNDAMENTAL Mathematician (a Video-taped lecture)

I strongly urge readers to watch my lecture, available in six parts from `YouTube`, and in two parts from `Vimeo`, see:

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/leon.html` .

That page contains links to both versions, as well as numerous input and output files for the Maple package `LEON`.

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