

Lehmer's Conjecture on the Non-vanishing of Ramanujan's Tau Function

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Abstract

In this paper we prove Lehmer's conjecture on Ramanujan's tau function, namely $\tau(n) \neq 0$ for each $n \geq 1$ by investigating the additive group structure attached to $\tau(n)$ with the aid of unique factorization theorem.

Note added Sept. 11, 2014 by the uploader, Doron Zeilberger: I was told that the present proof is incomplete. Will Lee informed me that he is currently working on fixing it.

¹ Let E_k ($k = 2, 4, \dots$) be the normalized Eisenstein series ($[4 : 108 - 122]$) given by

$$E_k = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \quad (1)$$

where $q := e^{i2\pi z}$ ($\Im(z) > 0$), B_k the Bernoulli number defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$$

and $\sigma_{k-1}(n)$ the divisor function:

$$\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}.$$

For an elliptic curve given by

$$y^2 = 4x^3 - g_2(z)x - g_3(z) \quad (2)$$

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where $g_2(z) = 120\zeta(4)E_4(z)$, $g_3(z) = 280\zeta(6)E_6(z)$ and $E_k(z)$ given by equation (1) and $\zeta(k)$ is Riemann zeta function:

$$\zeta(k) := \sum_{n=1}^{\infty} \frac{1}{n^k}.$$

A simple calculation ([1 : 14], [4 : 112]) shows the discriminant $\Delta(z) := 4^4(x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2$, where x_1, x_2 and x_3 are the roots the right side of equation (2), is given by

$$\Delta(z) = g_2(z)^3 - 27g_3(z)^2 = \frac{(2\pi)^{12}}{1728}(E_4(z)^3 - E_6(z)^2). \quad (3)$$

On the other hand Jacobi's theorem ([4 : 122]) asserts that

$$(2\pi)^{-12}\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}. \quad (4)$$

From equation (4), Ramanujan has defined his tau function $\tau(n)$ ([1], [2], [3], [4 : 122], [5] – [7]) by

$$q \prod_{n=1}^{\infty} (1 - q^n)^{24} := \sum_{n=1}^{\infty} \tau(n)q^n. \quad (5)$$

Notice that each $\tau(n)$ ($n \geq 1$) has an integer value. In a series of papers ([5] – [7]), D.H. Lehmer investigated the properties of $\tau(n)$ for $n \leq 300$, proved that $\tau(n) \neq 0$ for $n < 3316799$, later for $n < 214928639999$ ([1 : 22]). He also showed that if $\tau(n) = 0$ then n must be a prime. He then conjectured, what is nowadays known as Lehmer's conjecture ([6]) that

$$\tau(n) \neq 0 \text{ for each } n \geq 1. \quad (6)$$

A simple calculation ([3 : 21 – 22], [4 : 122 – 123]) shows

$$\tau(n) = \frac{65}{756}\sigma_{11}(n) + \frac{691}{756}\sigma_5(n) - \frac{691}{3} \sum_{j=1}^{n-1} \sigma_5(j)\sigma_5(n-j). \quad (7)$$

Since Lehmer's conjecture is equivalent to $3\tau(n) \neq 0$ for each $n \geq 1$, we write

$$A(n) := \frac{65}{252}\sigma_{11}(n) + \frac{691}{252}\sigma_5(n); B(n) := 691 \sum_{j=1}^{n-1} \sigma_5(j)\sigma_5(n-j). \quad (8)$$

Then $3\tau(n) = A(n) - B(n)$. Observe that $A(n)$ takes on integer value for each $n \geq 1$ since both $\tau(n)$ and $B(n)$ do. Now Lehmer's conjecture is, in view of equations (7), (8) and the unique factorization theorem, equivalent to:

$$A(n) \neq B(n) \text{ for each } n \geq 1. \quad (9)$$

Recent calculation by Bosman confirms Lehmer's conjecture for $n \leq 22798241520242687999$.

In this paper we prove equation (9) by showing that $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{k=0}^{q-1}$ forms an additive group of order q modulo q for $q \mid A(p)$, $q > p$, $p \equiv -1 \pmod{691}$, $[a_{i,k}]_{0 \leq i,k \leq q-1}$ $q \times q$ -matrix, with the aid of the unique factorization theorem, the pigeonhole principle and the remainder theorem. We prove equation (9) first for prime p then for p^α , $\alpha \geq 2$ and finally for any composite number n . Since $11 \nmid 690$ and since $(p+1) \mid (p^{11}+1)$, the following Lemma 1 evidently holds.

Lemma 1 *Let $A(p)$ be given by equation (8). Then the following two conditions (10) and (11) are equivalent:*

$$A(p) \equiv 0 \pmod{691}. \quad (10)$$

$$p \equiv -1 \pmod{691}. \quad (11)$$

If $691 \nmid A(p)$ or equivalently p does not satisfy equation (11) then we trivially have $A(p) \neq B(p)$ by equation (8). It suffices therefore to prove Lehmer's conjecture for prime p satisfying equation (10) or (11). In what follows, prime p satisfies either equation (10)

or (11). We first prove:

Lemma 2 *Let p satisfy equation (10) or (11). Then $A(p)$ has at least one prime factor q greater than p .*

Proof. Write $A(p)$ from equation (8) as

$$\begin{aligned} A(p) &= \frac{65}{252}(1 + p^{11}) + \frac{691}{252}(1 + p^5) \\ &= 3 + K_5 p^5 \end{aligned} \tag{12}$$

where $K_5 := \frac{691}{252} + \frac{65}{252}p^6$ is an integer with $p^5 < K_5 < p^6$ since $A(p)$ is an integer with $p^{10} < A(p) < p^{11}$. Suppose $A(p)$ has no prime factor greater than p . $A(p)$ then is written via the unique factorization theorem as

$$A(p) = 2^{e_0} q_1^{e_1} \dots q_m^{e_m}, \quad q_i < p, \quad e_i \geq 1 \quad (1 \leq i \leq m). \tag{13}$$

Notice that $A(p)$ has an even factor 2^{e_0} by substituting equation (11) into equation (8). Write $x = [x] + \{x\}$ where $[x]$ represents the integral part of x and $\{x\}$ the nonintegral part of x . Since $p^{10} < A(p) < p^{11}$, the representation for $A(p)$ in the base p is uniquely given from equation (13) by

$$A(p) = \sum_{i=0}^{10} b_i p^i, \quad b_i := [p \{ \frac{A(p)}{p^{i+1}} \}] \quad (0 \leq i \leq 10). \tag{14}$$

We show that each $b_i \neq 0$ ($0 \leq i \leq 10$). Indeed we prove

$$b_i \geq 5 \quad (1 \leq i \leq 10), \quad b_0 \neq 0. \tag{15}$$

The fact that $b_0 \neq 0$ follows from $p \nmid A(p)$. Likewise $b_{10} \geq 5$ follows from $p \nmid A(p)$ and $b_{10} \geq [\frac{65}{252}p] \geq [\frac{p}{252}] \geq 5$ in view of equations (14) and (17), where $p \geq 1381$. From equation (8) or (12), we readily have

$$\frac{65}{252}p^{11} < A(p) < (\frac{65}{252} + 3p^{-6})p^{11}. \tag{16}$$

Equation (16) is equivalent to

$$\frac{65}{252}p^{10-i} < \frac{A(p)}{p^{i+1}} < \left(\frac{65}{252} + 3p^{-6}\right)p^{10-i} \quad (1 \leq i \leq 9). \quad (17)$$

Write

$$65p^{10-i} = 252Q_i + R_i, \quad Q_i \geq 1, \quad 1 \leq R_i \leq 251 \quad (1 \leq i \leq 9). \quad (18)$$

Substitution of equation (18) into equation (17) reveals

$$Q_i + \frac{R_i}{252} < \frac{A(p)}{p^{i+1}} < \left(Q_i + \frac{R_i}{252} + 3p^{4-i}\right) \quad (1 \leq i \leq 9). \quad (19)$$

Inequality (19) implies $\left\{\frac{A(p)}{p^{i+1}}\right\} \geq \frac{R_i}{252} \geq \frac{1}{252}$ ($1 \leq i \leq 9$) and hence we have since $p \geq 1381$

$$b_i = \left[p\left\{\frac{A(p)}{p^{i+1}}\right\}\right] \geq \left[\frac{p}{252}\right] \geq 5 \quad (1 \leq i \leq 9). \quad (20)$$

This establishes inequality (15). Rewrite equation (14) as

$$A(p) = L_5p^5 + \sum_{i=0}^4 b_i p^i \quad (21)$$

where $L_5 := \sum_{i=0}^5 b_{5+i}p^i$. Subtraction of equation (21) from equation (12) with rearrangement of terms leads us to

$$\begin{aligned} p^5 &\leq (K_5 - L_5)p^5 \\ &= (b_0 - 3) + \sum_{i=1}^4 b_i p^i \\ &< (p - 1) \sum_{i=0}^4 p^i \\ &= p^5 - 1. \end{aligned} \quad (22)$$

Since each $b_i \geq 1$ ($0 \leq i \leq 10$) from equation (15), $K_5 - L_5 \geq 1$ follows from the the second line equality of equation (22) regardless of the value of $b_0 \neq 0$. Since $1 \leq b_i \leq p - 1$ ($0 \leq i \leq 10$) from equation (15) and since $b_0 - 3 < p - 1$, the third line inequality follows. Inequality (22) is absurd. Consequently the assumption that $A(p)$ has no prime factor $> p$ is false. This establishes Lemma 2.

It is easy to check our proof for Lemma 2 works for all primes $p > 252$ with inequality (15) replaced by $1 \leq b_i$ ($0 \leq i \leq 10$). A simple computation reveals Lemma 2 also holds for primes $p \leq 252$. Consequently Lemma 2 holds for all primes p . Thus the assumption that the prime p generated by equation (11) in Lemma 2 is redundant.

Let q be an odd prime prime factor of $A(p)$ greater than p . Existence of such a prime q is guaranteed by Lemma 2. Construct matrix $[a_{i,k}]_{0 \leq i, k \leq q-1}$ as follows:

$$a_{i,k} := \sum_{\substack{j=1 \\ i691\sigma_5(j)\sigma_5(p-j) \equiv k \pmod{q}}}^{p-1} 1. \quad (23)$$

Since $\sigma_5(j)\sigma_5(p-j) = \sigma_5(p-j)\sigma_5(p-(p-j))$, we have from equation (23)

$$a_{i,k} = 2 \sum_{\substack{j=1 \\ i691\sigma_5(j)\sigma_5(p-j) \equiv k \pmod{q}}}^{(p-1)/2} 1. \quad (24)$$

Then the matrix $[a_{i,k}]_{0 \leq i, k \leq q-1}$ has the following properties:

$$a_{i,k} \equiv 0 \pmod{2} \quad (0 \leq i, k \leq q-1). \quad (25)$$

$$a_{0,0} = p-1, \quad a_{0,k} = 0 \quad (1 \leq k \leq q-1). \quad (26)$$

$$a_{i,0} = a_{j,0} \quad (1 \leq i \neq j \leq q-1). \quad (27)$$

$$a_{i,k} = a_{q-i, q-k} \quad (1 \leq i, k \leq q-1). \quad (28)$$

$$i691 \sum_{j=1}^{p-1} \sigma_5(j)\sigma_5(p-j) \equiv \sum_{k=1}^{q-1} ka_{i,k} \pmod{q} \quad (1 \leq i \leq q-1). \quad (29)$$

$$\sum_{k=1}^{q-1} ka_{i,k} \equiv i \sum_{k=1}^{q-1} ka_{1,k} \pmod{q} \quad (1 \leq i \leq q-1). \quad (30)$$

Notice that given $a_{1,k}$ ($1 \leq k \leq q-1$), $a_{i,k}$ ($2 \leq i \leq q-1, 1 \leq k \leq q-1$) are reshuffles of $a_{1,k}$ ($1 \leq k \leq q-1$) and vice versa determined by

$$a_{i,k} = a_{1,i^{-1}k \bmod q} \iff a_{1,k} = a_{i,ik \bmod q} \quad (1 \leq i, k \leq q-1). \quad (31)$$

For each $i = 1, 2, \dots, q-1$, write $f_{j_i} := i691\sigma_5(j)\sigma_5(p-j) \bmod q$. Then $f_{j_i} = f_{p-j_i}$ ($1 \leq i \leq q-1$) from equation (23). Define for each $i = 1, 2, \dots, q-1$:

$$\begin{aligned} S_{i,l} &:= \{k : a_{i,k} = 2l\} \quad (1 \leq k \leq q, 1 \leq l \leq q_0) \\ \iff &= \{(j_{i_1}, j_{i_2}, \dots, j_{i_l}) : f_{j_{i_1}} = f_{j_{i_2}} = \dots = f_{j_{i_l}} = k\} \quad (1 \leq j_{i_1} < j_{i_2} < \dots < j_{i_l} \leq \frac{p-1}{2}) \\ S_{i,l} &= \emptyset \quad (1 \leq l \leq q_0) \text{ for } l > q_0. \end{aligned} \quad (32)$$

Since $q > p$ and since $a_{i,k}$ ($0 \leq i, k \leq q-1$) cannot be too large even number from equations (23) and (24), a positive integer $q_0 < q-1$ exists, depending on p and q , satisfying the last line of equation (32). It is clear from equation (32) with the aid of equation (31) that for each $l = 1, 2, \dots, q_0$:

$$S_{i,l} = S_{j,l} \quad (1 \leq i < j \leq q-1). \quad (33)$$

For each $q \mid A(p)$ with $q > p$, we then have from equations (31) – (33) that

$$\sum_{l=0}^{q_0} |S_{i,l}| = q \quad (1 \leq i \leq q-1). \quad (34)$$

$$\sum_{k=1}^{q-1} a_{i,k} = \sum_{l=1}^{q_0} 2l |S_{i,l}| = p-1 \quad (1 \leq i \leq q-1). \quad (35)$$

Equation (35) reads when $q \mid A(p)$ with $q < p$ that:

$$\sum_{k=0}^{q-1} a_{i,k} = p-1 \quad (1 \leq i \leq q-1). \quad (36)$$

Equations (25) – (31) readily follow from equations (23) and (24). Equation (29) is a restatement of the remainder theorem in view of equations (23), (32) and (35). Equations (34), (35) and (36) follow from the pigeonhole principle. Since we exclusively use $S_{1,l}$ ($1 \leq l \leq q_0$) in what follows, we show the following inequality:

$$|S_{1,l-1}| > l |S_{1,l}| \quad (2 \leq l \leq q_0). \quad (37)$$

To prove inequality (37) we use the second line of equation (32) for the definition of $S_{1,l}$.

Consider the map $\beta : S_{1,l} \mapsto S_{1,l-1} \times S_{1,l-1} \times \cdots \times S_{1,l-1}$ given by

$$\beta(j_1, j_2, \dots, j_l) := ((\beta_1(j_1), \beta_2(j_1), \dots, \beta_{l-1}(j_1)), (\beta_1(j_2), \beta_2(j_2), \dots, \beta_{l-1}(j_2)), \dots, (\beta_1(j_l), \beta_2(j_l), \dots, \beta_{l-1}(j_l))) \quad (38)$$

such that for each $i = 1, 2, \dots, l$:

$$\begin{aligned} \beta_1(j_i) &:= \min_{j_{i_k}} \{ |j_i - j_{i_k}| : a_{1,j_{i_k}} = 2(l-1) \} \\ f_{\beta_1(j_i)} &= f_{\beta_2(j_i)} = \cdots = f_{\beta_{l-1}(j_i)}. \end{aligned} \quad (39)$$

In the second line of equation (39), $\beta_k(j_i)$ ($2 \leq k \leq l-1, 1 \leq i \leq l$) are uniquely determined once $\beta_1(j_i)$ ($1 \leq i \leq l$) is determined by the first line of equation (39). Observe that $(\beta_1(j_i), \beta_2(j_i), \dots, \beta_{l-1}(j_i)) \in S_{1,l-1}$ ($1 \leq i \leq l$) are distinct from equations (32) and (39). To show that the map β given by equation (38) maps $S_{1,l}$ into a proper subset of $S_{1,l-1}$, write

$$\begin{aligned} \beta_1(j'_1) &:= \min_{j'_{1_k}} \{ |j_1 - j'_{1_k}| : a_{1,j'_{1_k}} = 2(l-1), \beta_1(j_1) \neq \beta_1(j'_1) \} \\ f_{\beta_1(j'_1)} &= f_{\beta_2(j'_1)} = \cdots = f_{\beta_{l-1}(j'_1)}. \end{aligned} \quad (40)$$

Observe that $(\beta_1(j'_1), \beta_2(j'_1), \dots, \beta_{l-1}(j'_1)) \in S_{1,l-1}$ and distinct from $(\beta_1(j_i), \beta_2(j_i), \dots, \beta_{l-1}(j_i))$ ($1 \leq i \leq l$) from equations (32), (39) and (40). Equations (39) and (40) imply that the map $\beta : S_{1,l} \mapsto S_{1,l-1} \times S_{1,l-1} \times \cdots \times S_{1,l-1}$ given by equation (38) maps $S_{1,l}$ into a proper subset of $S_{1,l-1}$ in a fashion of 1 to l . This establishes inequality (37). See Table 1 for examples of primes p with $q \mid A(p)$, $q > p$, satisfying inequality (37), where $q_0 \leq 4$.

Lehmer's conjecture therefore is equivalent via equation (29) for $i = 1$ to:

$$\sum_{k=0}^{q-1} k a_{1,k} \not\equiv 0 \pmod{q}. \quad (41)$$

Since both $A(p)$ and $B(p)$ are even and divisible by 691, we have $(A(p), B(p)) \geq 1382$. Suppose q divides both $A(p)$ and $B(p)$. Then by equation (29), we have:

$$\sum_{k=0}^{q-1} ka_{i,k} \equiv 0 \pmod{q} \quad (0 \leq i \leq q-1). \quad (42)$$

Clearly equation (42) is equivalent by equation (30) to:

$$\sum_{k=0}^{q-1} ka_{1,k} \equiv 0 \pmod{q}. \quad (43)$$

Since $\sum_{k=0}^{q-1} ka_{0,k} = 0 \equiv 0 \pmod{q}$ by equation (26), it follows that $\{\sum_{k=0}^{q-1} ka_{i,k} \pmod{q}\}_{i=0}^{q-1} = \{0\}$, the trivial additive group modulo q . Conversely, equation (42) or (43) implies both $q \mid A(p)$ and $q \mid B(p)$ by equation (29). On the other hand, since nonzero $a_{i,k}$ ($0 \leq i \leq q-1$) is even and ≥ 2 from equation (25), with the aid of the unique factorization theorem, equation (42) or (43) is equivalent to:

$$\min_{1 \leq i < j \leq q-1} \left(\sum_{k=0}^{q-1} ka_{i,k}, \sum_{k=0}^{q-1} ka_{j,k} \right) = 2q. \quad (44)$$

Consequently equation (42), (43) or (44) completely characterizes common odd prime factors of both $A(p)$ and $B(p)$. We thus have:

Lemma 3 *The following conditions are equivalent:*

(i) q divides both $A(p)$ and $B(p)$.

(ii) $\sum_{k=0}^{q-1} ka_{i,k} \equiv 0 \pmod{q}$ ($0 \leq i \leq q-1$).

$$(iii) \sum_{k=0}^{q-1} ka_{1,k} \equiv 0 \pmod{q}.$$

$$(iv) \left\{ \sum_{k=0}^{q-1} ka_{i,k} \pmod{q} \right\}_{i=0}^{q-1} = \{0\}, \text{ the trivial additive group modulo } q.$$

$$(v) \min_{1 \leq i < j \leq q-1} \left(\sum_{k=0}^{q-1} ka_{i,k}, \sum_{k=0}^{q-1} ka_{j,k} \right) = 2q.$$

Lemma 4 (Main Lemma) *Let p satisfy equation (10) or (11) and let $q \mid A(p)$ with $q > p$.*

Then $\left\{ \sum_{k=0}^{q-1} ka_{i,k} \pmod{q} \right\}_{i=0}^{q-1}$ forms an additive group of order q modulo q .

Proof. Let $a_{i,k}$ ($0 \leq i, k \leq q-1$) be defined by equation (23). We have for each $i = 1, 2, \dots, q-1$:

$$\begin{aligned} & \sum_{k=0}^{q-1} ka_{i,k} && + \sum_{k=0}^{q-1} ka_{q-i,k} \\ = & \sum_{k=1}^{q-1} ka_{i,k} && + \sum_{k=1}^{q-1} ka_{i,q-k} && \text{by (28)} \\ = & \sum_{k=1}^{q-1} ka_{i,k} && + \sum_{k=1}^{q-1} (q-k)a_{i,k} \\ = & q \sum_{k=1}^{q-1} a_{i,k} \\ = & q \sum_{l=1}^{q_0} 2l \mid S_{1,l} \mid && \text{by (35)} \\ = & q(p-1) && \text{by (35)} \end{aligned} \tag{45}$$

Notice that equation (45) holds regardless of $\left\{ \sum_{k=0}^{q-1} ka_{i,k} \pmod{q} \right\}_{i=0}^{q-1}$ being trivial or not.

We claim that $\left\{ \sum_{k=0}^{q-1} ka_{i,k} \right\}_{i=0}^{q-1}$ are all distinct. To show the claim observe that $\{S_{1,l}\}_{l=0}^{q_0}$ are disjoint from equation (32). Since $a_{i,k} = a_{1,i^{-1}k \pmod{q}}$ from equation (31), we have for each $l = 1, 2, \dots, q_0$:

$$\begin{aligned} \sum_{k \in S_{1,l}} ka_{i,k} &= \sum_{k \in S_{1,l}} ka_{1,i^{-1}k \pmod{q}} = \\ \sum_{k \in S_{1,l}} ik \pmod{q} a_{1,k} &= 2l \sum_{k \in S_{1,l}} ik \pmod{q}. \end{aligned} \tag{46}$$

It is evident for each $1 \leq i \neq j \leq q-1$ and each l ($1 \leq l \leq q_0$) that:

$$\sum_{k \in S_{1,l}} ik \pmod{q} \neq \sum_{k \in S_{1,l}} jk \pmod{q}. \tag{47}$$

For each $1 \leq i \neq j \leq q-1$, conjunction of equations (46) and (47) leads us to

$$\begin{aligned}
&= \sum_{k=0}^{q-1} ka_{i,k} \\
&= \sum_{l=1}^{q_0} 2l \sum_{k \in S_{1,l}} ik \pmod{q} \text{ by (46)} \\
&\neq \sum_{l=1}^{q_0} 2l \sum_{k \in S_{1,l}} jk \pmod{q} \text{ by (37) \& (47)} \\
&= \sum_{k=0}^{q-1} ka_{j,k} \text{ by (46)}.
\end{aligned} \tag{48}$$

Equation (48) establishes the claim. Since $\sum_{k=0}^{q-1} ka_{1,k} \pmod{q}$ is a generator for the additive group $\{\sum_{k=0}^{q-1} ka_{i,k} \pmod{q}\}_{i=0}^{q-1}$ from equation (30) if it is nontrivial, it suffices therefore to show that

$$\sum_{k=0}^{q-1} ka_{1,k} \not\equiv 0 \pmod{q}. \tag{49}$$

Write

$$C_i := \sum_{k=0}^{q-1} ka_{i,k} \quad (1 \leq i \leq q-1). \tag{50}$$

Notice that $\{C_i\}_{i=1}^{q-1}$ are distinct from equation (48). Rename C_i ($1 \leq i \leq q-1$) again as C_i ($1 \leq i \leq q-1$) in ascending order as follows:

$$C_1 < C_2 < \dots < C_{q-1}. \tag{51}$$

We claim that there is at least one pair $\{C_j, C_{j+1}\}$ ($1 \leq j \leq q-2$) from equation (51) such that

$$C_{j+1} - C_j < q-1 \text{ for some } j \text{ (} 1 \leq j \leq q-2 \text{)}. \tag{52}$$

Assume equation (52) is false. We then have:

$$\begin{aligned}
&C_{q-1} \\
&:= \max_{1 \leq i \leq q-1} \sum_{k=1}^{q-1} ka_{i,k} \text{ by (51)} \\
&:= \sum_{k=1}^{q-1} ka_{i_0,k} \text{ for some } i_0 \text{ (} 1 \leq i_0 \leq q-1 \text{)} \\
&= C_1 + \sum_{k=1}^{q-2} (C_{k+1} - C_k) \\
&\geq C_1 + \sum_{k=1}^{q-2} (q-1) \text{ by assumption} \\
&= C_1 + (q-2)(q-1) \\
&> (q-2)(q-1).
\end{aligned} \tag{53}$$

On the other hand, we estimate C_{q-1} from equations (23) and (46). Since each nonzero $a_{i_0,k}$ ($0 \leq i_0 \leq q-1$) is even ≥ 2 from equation (25), there are at most $(p-1)/2$ -numbers of nonzero $a_{i_0,k} \geq 2$ ($0 \leq k \leq q-1$). Notice that each nonzero $a_{i_0,k}$ is a small even number due to equations (32) and (35) with $2 \leq a_{i_0,k} \leq 2q_0$ ($0 \leq k \leq q-1$). It follows that there are at least $(q - (p-1)/2)$ -numbers of $a_{i_0,k} = 0$ ($0 \leq k \leq q-1$). We then have:

$$\begin{aligned}
& C_{q-1} \\
&= \sum_{k=0}^{q-1} k a_{i_0,k} \\
&= \sum_{k=0}^{q-1} i_0 k \pmod{q} a_{1,k} \quad \text{by (31)} \\
&= \sum_{l=1}^{q_0} 2l \left(\sum_{k \in S_{1,l}} i_0 k \pmod{q} \right) \text{ by (46)} \\
&< (\sum_{l=1}^{q_0} 2l |S_{1,l}|)(q-1) \\
&= (p-1)(q-1) \quad \text{by (35)} \\
&< (q-2)(q-1).
\end{aligned} \tag{54}$$

In the last line of inequality (54), we use the assumption $p+1 < q$ and hence $p-1 < q-2$. The last line of inequality (54) contradicts inequality (53). This establishes inequality (52). For j chosen from inequality (52), since each nonzero $a_{i,k} \geq 2$ ($1 \leq i \leq q-1$, $0 \leq k \leq q-1$), we then have:

$$2 \leq (C_j, C_{j+1}) = (C_j, C_{j+1} - C_j) < q-1. \tag{55}$$

Equation (55) implies $C_j := \sum_{k=0}^{q-1} k a_{u,k}$ and $C_{j+1} := \sum_{k=0}^{q-1} k a_{v,k}$ for some u, v ($1 \leq u, v \leq q-1$), have no common factor q , which leads to $q \nmid \sum_{k=0}^{q-1} k a_{1,k}$ in view of equation (30), thereby proving equation (49). Consequently, each $\sum_{k=0}^{q-1} k a_{i,k}$ ($1 \leq i \leq q-1$) has no factor q from equations (30) and (49). We thus have:

$$\sum_{k=0}^{q-1} k a_{i,k} \not\equiv 0 \pmod{q}, \quad 1 \leq i \leq q-1. \tag{56}$$

Equation (56) is equivalent that the map:

$$\left\{ \sum_{k=0}^{q-1} k a_{i,k} \pmod{q} \right\}_{i=0}^{q-1} \longmapsto \mathbb{Z}/q\mathbb{Z}$$

is an isomorphism. Furthermore equations (45) and (56) reveal the structure of the additive group $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{i=0}^{q-1}$ which is nontrivial, namely

$$\sum_{k=0}^{q-1} ka_{i,k} + \sum_{k=0}^{q-1} ka_{q-i,k} \equiv 0 \bmod q, \quad 1 \leq i \leq q-1. \quad (57)$$

Equations (56) and (57) show $\sum_{k=0}^{q-1} ka_{i,k} \bmod q$ and $\sum_{k=0}^{q-1} ka_{q-i,k} \bmod q$ are additive inverse to each other modulo q for each $i = 1, 2, \dots, q-1$. Clearly $\sum_{k=0}^{q-1} ka_{0,k} = 0 \equiv 0 \pmod{q}$ is the additive identity modulo q from equation (26). This completes the proof of Lemma 4.

Since $p \nmid A(p)$ from equation (12), conjunction of Lemma 3 and Lemma 4 leads us to:

Corollary 5 *Let $691 \mid A(p)$. An odd prime q divides both $A(p)$ and $B(p)$ only if $q < p$.*

From Lemma 4, we have in particular for $i = 1$:

$$B(p) = 691 \sum_{j=1}^{p-1} \sigma_5(j)\sigma_5(p-j) \equiv \sum_{k=0}^{q-1} ka_{1,k} \not\equiv 0 \bmod q \text{ by (29) \& (56)}. \quad (58)$$

Equation (58) implies $q \nmid B(p)$ and hence $A(p) \neq B(p)$ and $\tau(p) = (A(p) - B(p))/3 \neq 0$ via the unique factorization theorem if $691 \mid A(p)$. If $691 \nmid A(p)$, then since $691 \mid B(p)$ from equation (8), we trivially have $A(p) \neq B(p)$ and $\tau(p) = (A(p) - B(p))/3 \neq 0$ via the unique factorization theorem in this case too. We thus have:

Theorem 6 $\tau(p) \neq 0$ for each prime p .

For $691 \mid A(p)$ and $q \mid A(p)$ with $q > p$, since $\{\sum_{k=0}^{q-1} ka_{i,k}\}_{i=0}^{q-1}$ are distinct from equation (48) and since $q \nmid \sum_{k=0}^{q-1} ka_{i,k}$ ($1 \leq i \leq q-1$) from Lemma 4, we have $\sum_{k=0}^{q-1} ka_{i,k} = 2^s t$ ($s \geq 1$, $t = \text{odd}$, $1 \leq i \leq q-1$), where $q \nmid t$ from Lemma 4. Since $q > p$, and since each nonzero $a_{i,k} \geq 2$ from equation (25), there is at least one i ($1 \leq i \leq q-1$) such

that $\sum_{k=0}^{q-1} ka_{i,k} = 2t$, $q \nmid t$. We thus have from Lemma 1 (statement (v)) with the aid of unique factorization theorem:

Corollary 7 *Suppose p satisfies equation (10) or (11). Let $q \mid A(p)$ with $q > p$. Then*

$$\min_{1 \leq i < j \leq q-1} \left(\sum_{k=0}^{q-1} ka_{i,k}, \sum_{k=0}^{q-1} ka_{j,k} \right) = 2.$$

Now let $\alpha \geq 2$. Then equations (10) and (11) are no longer equivalent. As in the case of $\alpha = 1$, since $A(p^\alpha) \equiv 3 \pmod{p^5}$ and $p^{11\alpha-1} < A(p^\alpha) < p^{11\alpha}$ from equation (8), an almost identical proof of Lemma 2 works for $\alpha \geq 2$, where in equation (14), the upper limit for the sum is replaced by $11\alpha - 1$. We thus have:

Lemma 8 *Let $691 \mid A(p^\alpha)$ for $\alpha \geq 2$. There is at least one prime $q \mid A(p^\alpha)$ with $q > p^\alpha$.*

For $q \mid A(p^\alpha)$, construct matrix $[a_{i,k}]_{0 \leq i, k \leq q-1}$ exactly the same way as in equation (23). Then properties (25) – (31), (33) – (37) hold with p replaced by p^α . Likewise almost identical proof of Lemma 4 works for $\alpha \geq 2$. We thus have:

Lemma 9 *Let $691 \mid A(p^\alpha)$ for $\alpha \geq 2$. Let $q \mid A(p^\alpha)$ with $q > p^\alpha$. Then $\{\sum_{k=0}^{q-1} ka_{i,k} \pmod{q}\}_{i=0}^{q-1}$ forms an additive group of order q modulo q .*

In particular for $i = 1$ from Lemma 9 and equation (29), we have for $\alpha \geq 2$

$$B(p^\alpha) = 691 \sum_{j=1}^{p^\alpha-1} \sigma_5(j) \sigma_5(p^\alpha - j) \equiv \sum_{k=0}^{q-1} ka_{1,k} \not\equiv 0 \pmod{q}. \quad (59)$$

Equation (59) implies $q \nmid B(p^\alpha)$ and hence $A(p^\alpha) \neq B(p^\alpha)$ and $\tau(p^\alpha) = (A(p^\alpha) - B(p^\alpha))/3 \neq 0$ by the unique factorization theorem. If $691 \nmid A(p^\alpha)$, since $691 \mid B(p^\alpha)$ from equation (8), we then trivially have $A(p^\alpha) \neq B(p^\alpha)$ and $\tau(p^\alpha) = (A(p^\alpha) - B(p^\alpha))/3 \neq 0$

via the unique factorization theorem in this case too. We thus have:

Theorem 10 $\tau(p^\alpha) \neq 0$ for each $\alpha \geq 2$.

Finally we show that $\tau(n) \neq 0$ for any positive integer n .

Theorem 11 (*Lehmer's Conjecture*) $\tau(n) \neq 0$ for each $n \geq 1$.

Proof. Since $\tau(1) = 1$, it suffices to prove the theorem when n is composite from Theorem 6 and Theorem 10. Write

$$n = p_0^{s_0} p_1^{s_1} \cdots p_u^{s_u}, \quad p_0 := 2, \quad s_0 \geq 0, \quad s_j \geq 1, \quad 1 \leq j \leq u.$$

Since $\tau(n)$ is multiplicative ([1 : 92 – 93], [2 : 52 – 53], [4 : 122], [5], [6]), Theorem 11 readily follows from Theorem 6 or Theorem 10 , namely

$$\begin{aligned} \tau(n) &= \prod_{j=0}^u \tau(p_j^{s_j}) \\ &\neq 0. \end{aligned} \tag{60}$$

This completes the proof.

Suppose for each $\alpha \geq 1$,

$$A(p^\alpha) \equiv 0 \pmod{691}. \tag{61}$$

Equation (61) is equivalent to:

$$p^{(\alpha+1)} \equiv 1 \pmod{691} \quad \text{and} \quad (p-1, 691) = 1. \tag{62}$$

Equation (62) implies the following periodicity theorem modulo 691:

Theorem 12 (*periodicity modulo 691*) Suppose $691 \mid A(p^\alpha)$ for $\alpha \geq 1$. Then we have:

$$A(p^{\alpha+k(\alpha+1)}) \equiv 0 \pmod{691}, \quad k = 0, 1, 2, \dots$$

The values of α satisfying the periodicity of $A(p^\alpha) \equiv 0 \pmod{691}$ for each $\alpha \geq 1$ have gaps in view of equation (62) and Fermat's little theorem, namely $A(p^\alpha) \not\equiv 0 \pmod{691}$ if and only if the factors of $\alpha + 1$ do not divide $690 = 2 \cdot 3 \cdot 5 \cdot 23$. Thus $A(p^\alpha) \not\equiv 0 \pmod{691}$ for α in the following set S of numbers:

$$S := \{6, 10, 12, 16, 18, 28, 30, 36, 40, 42, 46, 48, 52, 58, \dots\}$$

Needless to say $A(p^\alpha) \neq B(p^\alpha)$ and hence $\tau(p^\alpha) \neq 0$ for each $\alpha \in S$ by equation (8) with the aid of the unique factorization theorem.

Remark 13 *If an odd prime $q \mid A(p^\alpha)$, $\alpha \geq 1$ with $q < p^\alpha$, as long as $\{\sum_{k=0}^{q-1} ka_{i,k} \pmod{q}\}_{i=0}^{q-1}$ forms an additive group of order q modulo q , then $q \nmid B(p^\alpha)$ by Lemma 4 or Lemma 9. It follows that $A(p^\alpha) \neq B(p^\alpha)$ and hence $\tau(p^\alpha) = (A(p^\alpha) - B(p^\alpha))/3 \neq 0$ in this case too. For $691 \mid A(p)$, computer simulation reveals $A(p)$ has at least one odd prime factor $q \neq 691$, $q \mid A(p)$ with $q < p$ for which $q \nmid B(p)$ for each prime $p \leq 1100000$ except $p = 186569, 290219, 464351, 671651$. Let $691 \mid A(p)$ and let $A_1(p)$ be the product of prime divisors $q \mid A(p)$ for which $q < p$ with their respective powers and $A_2(p)$ the product of prime divisors $q \mid A(p)$ for which $q > p$ with their respective powers. Computer simulation shows $C_1 p^2 < A_1(p) < C_2 p^5$ and $C_3 p^6 < A_2(p) < C_4 p^{10}$ with absolute constants $C_1, C_2, C_3, C_4 < 1$ for primes $p \leq 1100000$.*

In Table 1, we list primes p such that both 691 and q divide $A(p)$ with $q > p$ and the cardinality $|S_{1,l}|$ ($1 \leq l \leq 5$), thereby confirming inequality (37) with $q_0 \leq 4$. Notice that in Table 1, each prime p with the associated prime $q \mid A(p)$ with $q > p$, satisfies equations (34) and (35). Computer simulation reveals that the majority of respective relatively large odd prime factors less than p of both $A(p)$ and $B(p)$ are distinct. Likewise an overwhelming majority of common odd prime factors of both $A(p)$ and $B(p)$ for which $691 \mid A(p)$ are relatively small apart from 691, thereby confirming Corollary 5. In Table 2, we list primes $p \leq 3000000$ such that $691 \mid A(p)$ and the odd prime factors of $(A(p), B(p))$ are ≥ 11 .

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Table 1

p	q	$ S_{1,0} $	$ S_{1,1} $	$ S_{1,2} $	$ S_{1,3} $	$ S_{1,4} $	$ S_{1,5} $
8291	216113	212008	4065	40	0	0	0
29021	1357091	1342657	14358	76	0	0	0
30403	1283839	1268731	15015	93	0	0	0
34549	789673	772578	16918	175	2	0	0
51133	112919	89995	20474	2267	174	9	0
53897	371549	345582	25014	925	28	0	0
96739	392957	347376	42917	2543	118	3	0

Table 2

p	$(A(p), B(p))$
547271	2.3.11.691
610843	2.3.17.691
988129	2.3.5.13.691
1112509	2.3.5.23.691
1336393	2.3.101.691
1405493	2.3.113.691
1716463	$2 \cdot 3^2 \cdot 23 \cdot 691$
1875373	$2 \cdot 23 \cdot 691$
1940327	$2^2 \cdot 3^2 \cdot 13 \cdot 691$
2126897	$2 \cdot 3^3 \cdot 19 \cdot 691$
2128279	$2^2 \cdot 5 \cdot 11 \cdot 691$

p	$(A(p), B(p))$
2161447	$2^2.23.691$
2198761	$2.43.691$
2447521	$2.23.691$
2479307	$2.23.691$
2538733	$2.11.691$
2542879	$2^4.3.5.23.691$
2956097	$2.23.691$

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