Lagrange Inversion Without Tears (Analysis) (based on Henrici)

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Def 1: A formal Laurent series (f.L.s) is
\[ \sum_{N}^{\infty} a_n x^n, \]
where \(a_n\) and \(x\) are symbols and \(N\) is an integer.

Def 2: \([x^n]f(x)\) is the coeff. of \(x^n\) in \(f(x)\).

Def 3: \(\text{Res}\ f(x)\) is the coeff. of \(x^{-1}\) in \(f(x)\).

Def 4: Given a sequence \(f_i\) of f.L.s. starting at \(N_i\), their sum \(\sum f_i\) makes sense if the \(N_i\) are bounded below and for every \(n\), the set \([x^n]f_i(x)\) has only finitely many non-zero terms, and then the coeff. of \(x^n\) in \(\sum f_i\) is by definition, that sum.

Def 5
\[ c x^N \sum_{M} a_n x^n := \sum_{M} c a_n x^{n+N} : \]
(convince yourself that that the rhs makes sense)

Def. 6 If \(f(x) = \sum_{N}^{\infty} a_n x^n\) is a f.L.s., and so is \(g(x)\) then
\[ f(x)g(x) := \sum_{N} a_n x^n g(x) \]
(convince yourself that that the rhs makes sense).

Def. 7
\[ (\sum_{N} a_n x^n)' := \sum_{N} n a_n x^{n-1} \]

Prop. 1 (Product Rule): \((f(x)g(x))' = f'(x)g(x) + f(x)g'(x)\)

Proof: True for \(f(x) = x^m, g(x) = x^n\), and hence in general, since both sides are linear in \(f(x)\), and in \(g(x)\).

Prop. 2: \((f(x)^k)' = kf(x)^{k-1}f'(x)\)

Proof: True for \(k = 1\), and then by induction on \(k\), for positive \(k\) and for negative \(k\) by using the product rule applied to \(1 = f(x)^k f(x)^{-k}\).

Prop. 3: (Chain Rule): If \(\Phi\) and \(f(x)\) are f.L.s. then \((\Phi(f(x)))' = \Phi'(f(x))f'(x)\).
Proof: True for $\Phi = x^k$ thanks to Prop.2, now extend by linearity.

Prop. 4: If $f(x)$ is a f.L.s. then $\text{Res}(f'(x)) = 0$.

Proof: $(x^n)' = nx^{n-1}$ can never be a multiple of $x^{-1}$, hence true for monomials, and by linearity for all $f(x)$.

Prop. 5 (Integration by parts) If $f(x)$ and $g(x)$ are f.L.s. then $\text{Res}(f'(x)g(x)) = -\text{Res}(f(x)g'(x))$.

Proof: By Prop. 1 and 4.

Prop. 6 (change of variables): Let $u(t)$ be a f.L.s starting at $t$ (i.e. $N = 1$) and $\Psi(z)$ be any f.L.s. then $\text{Res}_t(u'(t)\Psi(u(t))) = \text{Res}_z\Psi(z)$.

Proof: By linearity enough to prove it for monomials $\Psi(z) = z^k$. Both sides are 0 if $k \neq -1$, the right by definition 3, the left by Prop. 4. When $k = -1$ the right is 1, by definition and the left is $\text{Res}(u'(t)/u(t)) = (u_1 + 2u_2t + \ldots)/(u_1 t + u_2 t^2 + \ldots) = 1/t + O(1)$.

Theorem (Lagrange Inversion Theorem): If $u(t)$ and $\Phi(t)$ are f.L.s. starting at $t$ and $t^0$ respectively, then $u(t) = t\Phi(u(t))$ implies \[ [t^n]u(t) = (1/n)[z^{n-1}]\Phi(z)^n \]

Proof:

\[ [t^n]u(t) = \text{Res}_t(u(t)t^{-n-1}) = \text{Res}_t(u(t)(t^{-n}/(-n))') = (1/n)\text{Res}_t(u(t)t^{-n}) \] given

\[ = (1/n)\text{Res}_z(\Phi(u(t))/u(t))^n \quad (6) = (1/n)\text{Res}_z(\Phi(z)^n/z^n) = (1/n)[z^{n-1}]\Phi(z)^n \quad \square \]

Added March 2, 2015: Generalizing, we have

Theorem (Generalized Lagrange Inversion Theorem): If $u(t)$ and $\Phi(t)$ are f.L.s. starting at $t$ and $t^0$ respectively, and $G(t)$ is yet another formal power series, then $u(t) = t\Phi(u(t))$ implies \[ [t^n]G(u(t)) = (1/n)[z^{n-1}]G'(z)\Phi(z)^n \]

Proof:

\[ [t^n]G(u(t)) = \text{Res}_t(G(u(t))t^{-n-1}) = \text{Res}_t(G(u(t))(t^{-n}/(-n))') = (1/n)\text{Res}_t(G'(u(t))u(t)t^{-n}) \] given

\[ = (1/n)\text{Res}_z(G'(t)G(u(t))(\Phi(u(t))/u(t))^n) \quad (6) = (1/n)\text{Res}_zG'(z)(\Phi(z)^n/z^n) = (1/n)[z^{n-1}]G'(z)\Phi(z)^n \quad \square \]