Solution of American Mathematical Monthly #11929 (Oct. 2016, Proposed by Donald Knuth)

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In [K], Donald Knuth proposed the following intriguing problem.

11929 *Proposed by Donald Knuth, Stanford University, Stanford, CA.* Let a_n be the number of ways in which a rectangular box that contains $6n$ squares tiles in three rows of length $2n$ can be split into two connected regions of size $3n$ without cutting any tiles. Thus $a_1 = 3, a_2 = 19, a_3 = 85$. Taking $a_0 = 1$, find a closed form for the generating function $A(z) = \sum_{n=0}^{\infty} a_n z^n$. What is the asymptotic nature of a_n as $n \to \infty$?

Ans.

$$A(z) = \frac{z\left(2\,z^6 + 15\,z^5 + 21\,z^4 - 38\,z^3 + 40\,z^2 - 9\,z + 1\right)}{(z-1)^3\left(z^2 + 4\,z - 1\right)^2} + \frac{(1-3z)^2}{(1-4z-z^2)^2} \cdot (1-4z)^{-\frac{1}{2}} \quad .$$

The asymptotic of a_n, to order five is:

$$a_n \asymp \frac{16}{\sqrt{\pi}} \cdot \frac{4^n}{\sqrt{n}} \cdot$$

$$\left(1 - \frac{169}{8}\,n^{-1} + \frac{113713}{128}\,n^{-2} - \frac{54662275}{1024}\,n^{-3} + \frac{136066657419}{32768}\,n^{-4} - \frac{103820177963271}{262144}\,n^{-5} + O(\frac{1}{n^6})\right) \quad .$$

Proof: We first need a **crucial lemma**

Lemma: Let $f(r, w)$ be the number of configurations (with the white and black areas both connected) with r columns and w white squares (and hence $3r - w$ black squares), then the bi-variate generating function

$$F(x, y) = \sum_{r=0}^{\infty} \sum_{w=0}^{3r} f(r, w) x^r y^w$$

is given by the *rational function*

$$F(x, y) = \frac{Top(x, y)}{Bot(x, y)} \quad ,$$

where the numerator, $Top(x, y)$, is given by

$$Top(x,y) = -3\,x^4y^2 - x^2y^6 + 4\,x^4y^4 + 12\,x^4y^7 + 8\,x^6y^7 + 25\,x^6y^8 + 8\,x^6y^{11} - 6\,x^2y - 6\,x^2y^2$$
$$+25\,x^6y^{10} - 6\,x^2y^4 - 6\,x^2y^5 - 3\,x^4y^{10} - 2\,x^4y^9 + 4\,x^4y^8 + 2\,x^4y^6 + 12\,x^4y^5 - 6\,x^2y^3$$
$$+x^4 - x^2 - 2\,x^4y^3 + 6\,x^6y^{12} + 28\,x^6y^9 + 10\,x^{12}y^{21} + 4\,x^{10}y^{21} - 5\,x^{12}y^{20} + 12\,x^{12}y^{22}$$
$$+4\,x^{12}y^{23} - 5\,x^{12}y^{16} + x^8y^8 - 28\,x^8y^{11} - 9\,x^8y^{10} - 42\,x^8y^{12} + 6\,x^6y^6$$
$$-6\,x^6y^5 - 6\,x^6y^{13} - 28\,x^8y^{13} - 9\,x^8y^{14} - 8\,x^6y^{14} - 7\,x^8y^6 + 32\,x^{10}y^{11} + 35\,x^{10}y^{12}$$
$$-8\,x^6y^4 - 2\,x^8y^7 - 2\,x^6y^{15} + x^8y^{16} - 25\,x^{10}y^{14} - 38\,x^{10}y^{15} + 12\,x^{10}y^{13} + 35\,x^{10}y^{18}$$

$$-28\,x^{12}y^{18}-22\,x^{12}y^{19}+32\,x^{10}y^{19}+12\,x^{10}y^{17}-25\,x^{10}y^{16}-2\,x^{6}y^{3}-7\,x^{8}y^{18}$$
$$-2\,x^{8}y^{17}-4\,x^{8}y^{19}+17\,x^{10}y^{20}-22\,x^{12}y^{17}+x^{4}y^{12}-x^{6}y^{16}+17\,x^{10}y^{10}$$
$$-x^{6}y^{2}-4\,x^{14}y^{21}-2\,x^{14}y^{20}$$
$$-2\,x^{14}y^{22}+4\,x^{10}y^{9}-4\,x^{8}y^{5}+12\,x^{12}y^{14}+10\,x^{12}y^{15}+4\,x^{12}y^{13} \quad ,$$

and the denominator, $Bot(x,y)$ is given by

$$2\,(x+1)^2\left(xy^2+1\right)(xy+1)\left(xy^3+1\right)^2\left(xy+xy^2+1\right)(x-1)^2\left(xy^2-1\right)\left(xy^3-1\right)^2(xy-1)\left(xy+xy^2-1\right) \quad .$$

Sketch of proof: We set-up a "grammar" with ten states, corresponding to how a column can be colored, and connectivity considerations, set-up a system of ten equations and ten unknowns, add the solutions up, and get the above monster rational function. □

Since $a_n = f(2n,3n)$, we have

$$A(z) = \sum_{n=0}^{\infty} f(2n,3n) z^n \quad ,$$

is a '*slanted diagonal*' of the rational function $F(x,y)$ of 'slope' $3/2$. By a famous theorem of Hillel Furstenberg [F], it is guaranteed to be an algebraic formal power series, that can be easily found by taking a certain residue, but we prefer to crank-out sufficiently many terms (40 suffice) of a_n, using $F(x,y)$, and fit them into an algebaric equation for $A(z)$, e.g. using Maple's `gfun[listoalgeq]` command. We get that $A(z) = A$ is a solution of the quadratic equation

$$-10\,z+1-110\,z^3+178\,z^5-618\,z^6+108\,z^{10}+862\,z^7-485\,z^8+88\,z^9+16\,z^{11}+46\,z^2+116\,z^4$$
$$-2\,z\,(4\,z-1)\left(2\,z^6+15\,z^5+21\,z^4-38\,z^3+40\,z^2-9\,z+1\right)(z-1)^3\,A$$
$$+\,(4\,z-1)\left(z^2+4\,z-1\right)^2(z-1)^6\,A^2 \;=0 \quad .$$

Using the quadratic formula, taking the right solution (that is a formal power series with positive coefficients), and simplifying, yields the closed form solution stated above for $A(z)$. Expanding $(1-4z)^{-1/2}$ by the binomial theorem, using Stirling, and performint routine manipulations (all done automatically by Maple) produces the stated asymptotic formula for a_n.

Supporting Maple code can be gotten from

`http://math.rutgers.edu/~zeilberg/tokhniot/DEKjs.txt` .

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References

[F] H. Furstenberg, *Algebraic functions over finite fields*, J. of Algebra **7** (1967), 271-277.

[K] Donald Knuth, *Proposed problem #11929*, Amer. Math, Monthly **123** (2016), p. 831.

[Z] Doron Zeilberger, *webpage of Rutgers University Experimental Mathematics (Math 640) Class, Fall 2016*,
`http://www.math.rutgers.edu/~zeilberg/math640_17.html` .

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