Proof of an Identity of Don Knuth

Doron ZEILBERGER

Don Knuth, in a letter to Richard Stanley, Cc’d to several people, asked whether anybody has seen the following identity before:

$$\sum_{0 \leq x_1, \ldots, x_n \leq 1} \frac{(z - x_1 z^{n+1}) (z^2 - x_2 z^{n+1}) \cdots (z^n - x_n z^{n+1}) z^{n-x_1-(n-1)x_2-\cdots-x_n}}{(x_1 + 1)(x_1 x_2 + x_2 + 1) \cdots (x_1 \ldots x_n + \cdots + x_n + 1)} = \frac{(1 - z)(1 - z^2) \cdots (1 - z^{n+1})}{(n + 1)! (1 - z)^{n+1}}.$$ 

I have not. Here is a proof.

I will prove the more general identity:

$$A(n, r) := \sum_{0 \leq x_1, \ldots, x_n \leq 1} \frac{(z - x_1 z^{n+1}) (z^2 - x_2 z^{n+1}) \cdots (z^n - x_n z^{n+1}) z^{n-x_1-(n-1)x_2-\cdots-x_n}}{(r x_1 + 1)(r x_1 x_2 + x_2 + 1) \cdots (r x_1 \ldots x_n + x_2 \ldots x_n + \cdots + x_n + 1)}$$

$$= \frac{(1 - z)(1 - z^2) \cdots (1 - z^n)}{(1 - z)^{n+1}} \sum_{i=1}^{n+1} \frac{z^{n-i+1}(1 - z)^{r!}}{(r + i - 1)! (n - i + 1)!}.$$ 

(*)

The original identity follows by plugging $r = 1$ and using the Binomial Theorem: $(z + (1 - z))^{n+1} - z^{n+1} = 1 - z^{n+1}$.

To prove the identity for $A(n, r)$, split the sum according to whether $x_1 = 0$ or $x_1 = 1$ to get the recurrence:

$$A(n, r) = z^n A(n - 1, 1) + \frac{(1 - z^n)}{r + 1} A(n - 1, r + 1).$$

Since $A(0, r) = 1$, and the expression on the right of (*) obviously also satisfies this recurrence, we are done. □

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1 Department of Mathematics, Temple University, Philadelphia, PA 19122, USA. zeilberg@math.temple.edu