Proof of an Identity of Don Knuth

Doron ZEILBERGER[1]

Don Knuth, in a letter to Richard Stanley, Cc-d to several people, asked whether anybody has seen the following identity before:

$$\sum_{0 \le x_1, \dots, x_n \le 1} \frac{(z - x_1 z^{n+1})(z^2 - x_2 z^{n+1}) \dots (z^n - x_n z^{n+1}) z^{-n x_1 - (n-1) x_2 - \dots - x_n}}{(x_1 + 1)(x_1 x_2 + x_2 + 1) \dots (x_1 \dots x_n + \dots + x_n + 1)}$$
$$= \frac{(1-z)(1-z^2)\dots(1-z^{n+1})}{(n+1)!\,(1-z)^{n+1}}.$$

I have not. Here is a proof.

I will prove the more general identity:

$$A(n,r) := \sum_{0 \le x_1, \dots, x_n \le 1} \frac{(z - x_1 z^{n+1})(z^2 - x_2 z^{n+1}) \dots (z^n - x_n z^{n+1}) z^{-n x_1 - (n-1) x_2 - \dots - x_n}}{(r x_1 + 1)(r x_1 x_2 + x_2 + 1) \dots (r x_1 \dots x_n + x_2 \dots x_n + \dots + x_n + 1)}$$
$$= \frac{(1-z)(1-z^2)\dots(1-z^n)}{(1-z)^{n+1}} \cdot \sum_{l=1}^{n+1} \frac{z^{n-l+1}(1-z)^l r!}{(r+l-1)!(n-l+1)!}. \qquad (*)$$

The original identity follows by plugging $r = 1$ and using the Binomial Theorem: $(z + (1-z))^{n+1} - z^{n+1} = 1 - z^{n+1}$.

To prove the identity for $A(n,r)$, split the sum according to whether $x_1 = 0$ or $x_1 = 1$ to get the recurrence:

$$A(n,r) = z^n A(n-1,1) + \frac{(1-z^n)}{r+1} A(n-1, r+1) \quad .$$

Since $A(0,r) = 1$, and the expression on the right of $(*)$ obviously also satisfies this recurrence, we are done. □

[1] Department of Mathematics, Temple University, Philadelphia, PA 19122, USA. zeilberg@math.temple.edu
http://www.math.temple.edu/~zeilberg ftp://ftp.math.temple.edu/pub/zeilberg .
Sept. 19, 1995.