PROOF OF A CONJECTURE OF GESSEL

MANUEL KAUERS * AND CHRISTOPH KOUTSCHAN * AND DORON ZEILBERGER †

Abstract. We present a computer proof for the empirical observation of Gessel that the number of closed paths in the quarter plane with steps going east, west, south-east, and north-west has a nice closed form.

1. Introduction

There is a certain family of lattice walks, let's call them the Gessel walks, whose counting function is puzzling the combinatorialists already for several years. A Gessel walk is a walk in the lattice $\mathbb{Z}^2$ that stays entirely in the first quadrant (viz. it is a walk in $\mathbb{N}^2$) and that only consists of unit steps chosen from $G := \{\leftarrow, \rightarrow, \swarrow, \nearrow\}$. If $f(n; i, j)$ denotes the number of Gessel walks with exactly n steps starting at the origin $(0, 0)$ and ending at the point (i, j), then the counting function is the multivariate power series

$$F(t; x, y) := \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} f(n; i, j) x^i y^j t^n.$$

We would call this power series holonomic (with respect to t) if it satisfies an ordinary linear differential equation (with respect to t) with polynomial coefficients in t, x, y. This may or may not be the case.

For example, the Kreweras walks are defined just like the Gessel walks, but with the unit steps chosen from $\{\leftarrow, \downarrow, \nearrow\}$ instead of from G. It is a classical result [?] that their counting function is holonomic. In constrast, Bousquet-Mélou and Petkovšek showed that the counting function for certain Knight walks is not holonomic [?]. Mishna [?] provides a systematic study of all the possible walks in the quarter plane with steps chosen from any step set $S \subseteq \{\leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow, \swarrow\}$ with $|S| = 3$. She shows that the counting functions for the step sets $\{\nearrow, \searrow, \nwarrow\}$ and $\{\nearrow, \searrow, \uparrow\}$ (and some others that are equivalent to those by symmetry) are not holonomic while all others are holonomic.

For the number of walks returning to the origin, there is sometimes a nice closed form representation, even if there is no such representation for the number of walks to an arbitrary point (i, j). For instance, if $k(n; i, j)$ denotes the number of Kreweras walks of length n from $(0, 0)$ to (i, j), then [?]

$$k(3n; 0, 0) = \frac{4^n}{(n+1)(2n+1)} \binom{3n}{n} \qquad (n \geq 0)$$

and $k(n; 0, 0) = 0$ if n is not a multiple of 3.

Gessel [?] observed that a similar representation seems to exist for the Gessel walks. He conjectured the following closed form representation.

Theorem 1. *Let $f(n; i, j)$ denote the number of Gessel walks going in n steps from $(0, 0)$ to (i, j). Then $f(n; 0, 0) = 0$ if n is odd and*

$$f(2n; 0, 0) = 16^n \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} \qquad (n \geq 0),$$

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where $(a)_n := a(a+1)\cdots(a+n-1)$ *denotes the Pochhammer symbol.*

The purpose of the present paper is to prove this theorem. We will do so by computing a recurrence in n for $f(n;0,0)$. Then the statement follows directly by verifying that the right hand side satisfies the same recurrence and that the initial values match. Our recurrence has order 32, polynomial coefficients of degree 172, and involves integers with up to 385 decimal digits. As this is somewhat too much to be printed here (it would cover about 250 pages), we provide it electronically at

http://www... [DORON: PLEASE PROVIDE THE RIGHT URL]

Our result implies that $F(t;0,0)$ is holonomic with respect to t, but it has no direct implications concerning the holonomy of $F(t;x,y)$ for other x, y of interest, e.g., $x = y = 1$, or even for general x, y. There is, however, strong evidence that even the general counting function $F(t;x,y)$ with "symbolic" x and y is holonomic in t, see [**?**].

2. The Quasi-Holonomic Ansatz

2.1. **Annihilating Operators.** Let S_n, S_i, S_j be the shift operators, acting on $f(n;i,j)$ in the natural way, e.g., $S_n f(n;i,j) = f(n+1;i,j)$. An annihilating operator of $f(n;i,j)$ is an operator R with

$$R(n,i,j,S_n,S_i,S_j)f(n;i,j) = 0.$$

Those operators belong to a noncommutative polynomial algebra $\mathbb{Q}(n,i,j)[S_n,S_i,S_j]$, and together they form a left ideal in that algebra, called the *annihilator* of $f(n;i,j)$.

Our goal is to find an annihilating operator for Gessel's $f(n;i,j)$ that implies the conjecture. For example, it would be sufficient to know an annihilating operator $R(n,i,j,S_n)$ free of the shifts S_i and S_j, because then $R(n,0,0,S_n)$ would be an annihilating operator for $f(n;0,0)$.

For this reasoning to apply, we could actually be less restrictive and allow also shifts in i and j to occur in R, as long as they disappear when i and j are set to zero. Our goal, therefore, is to find operators P, Q_1, Q_2 such that

$$R(n,i,j,S_n,S_i,S_j) = P(n,S_n) + iQ_1(n,i,j,S_n,S_i,S_j) + jQ_2(n,i,j,S_n,S_i,S_j)$$

annihilates $f(n;i,j)$. This is the *quasi-holonomic ansatz* [**?**].

2.2. **Discovering annihilating operators.** We search for annihilating operators by making, for some fixed d, an ansatz

$$R = \sum_{0\leq e_1,\dots,e_6\leq d} c_{e_1,\dots,e_6} n^{e_1} i^{e_2} j^{e_3} S_n^{e_4} S_i^{e_5} S_j^{e_6}$$

with undetermined coefficients $c_{e_1,\dots,e_6}$. Applying this "operator template" to $f(n;i,j)$ gives

$$\sum_{0\leq e_1,\dots,e_6\leq d} c_{e_1,\dots,e_6} n^{e_1} i^{e_2} j^{e_3} f(n+e_4;i+e_5,j+e_6),$$

which, when equated to zero for any specific choice of n, i, j yields a linear constraint for the undetermined coefficients. (Note that $f(n;i,j)$ can be computed efficiently for any given $n,i,j \in \mathbb{Z}$.)

By taking several different n, i, j we obtain a linear system of equations. If that system has no solution, then there is definitely no annihilating operator matching the template. If there are solutions, then these are candidates for annihilating operators.

We can clearly restrict the search to quasi-holonomic operators by leaving out unwanted terms in the ansatz for R.

2.3. **Verifying conjectured annihilating operators.** An algorithm was given in [**?**] for deciding whether some given operator $R \in \mathbb{Q}(n,i,j)[S_n,S_i,S_j]$ annihilates $f(n;i,j)$ or not. We repeat this algorithm for the sake of self-containedness.

First note that the step set $\{\leftarrow, \rightarrow, \nearrow, \swarrow\}$ gives readily rise to the recurrence

$$f(n+1;i,j) = f(n;i+1,j) + f(n;i-1,j) + f(n;i+1,j+1) + f(n;i-1,j-1),$$

and therefore the "trivial operator"

$$T := S_n S_i S_j - S_i^2 S_j - S_j - S_i^2 S_j^2 - 1$$

certainly annihilates $f(n;i,j)$. Instead of checking that R annihilates $f(n;i,j)$, we will check that TR annihilates $f(n;i,j)$. By the following lemma, this is sufficient.

Lemma 1. *Suppose that an operator R is such that $(TR)f(n;i,j) = 0$. Then it can be checked whether $Rf(n;i,j) = 0$.*

Proof. $(TR)f(n;i,j) = 0$ implies that T annihilates $Rf(n;i,j)$, i.e., $Rf(n;i,j)$ also satisfies the above recurrence. Therefore, in order to show that $Rf(n;i,j) = 0$ entirely, it suffices to show that $Rf(n;i,j) = 0$ for $n = 0$ and all i and j. If r_n bounds the degree of S_n in R, then it suffices to verify $Rf(n;i,j) = 0$ for $n = 0$ and $0 \leq i,j \leq r_n$, because we clearly have $f(n;i,j) = 0$ for $i > n$ or $j > n$. This leaves us with checking finitely many values, which can be done. □

By the lemma, in order to check $Rf(n;i,j) = 0$, it suffices to be able to check $(TR)f(n;i,j) = 0$. For checking the latter, compute operators U, V with $TR = UT + V$ by division with remainder. Then

$$(TR)f(n;i,j) = 0 \iff Vf(n;i,j) = 0.$$

If V is the zero operator, then we are done, otherwise we proceed recursively to show that $Vf(n;i,j) = 0$ (compute U', V' with $TV = U'T + V'$, observe that $(TV)f(n;i,j) = 0$ iff $V'f(n;i,j) = 0$, and so on.) As T has constant coefficients and, for any $d > 0$, the commutation rules in $\mathbb{Q}(n,i,j)[S_n,S_i,S_j]$ are such that $S_i n^d = n^d S_i$, $S_j n^d = n^d S_j$ and $S_n n^d = n^d S_n + \mathrm{O}(n^{d-1})$ (and similarly for i and j in place of n), it follows that the degree of V with respect to n, i, j will be strictly smaller than the degree of R with respect to these variables. Therefore, the recursion must eventually come to an end.

2.4. **Nice idea, but...** At this point, we know that all we need for proving the conjecture is a quasi-holonomic annihilating operator for $f(n;i,j)$. We know how to search for such operators, and once empirically discovered, we know how to verify them.

Unfortunately, it turned out that if a quasi-holonomic annihilating operator for $f(n;i,j)$ exists at all, then it must be quite large. It was shown [**?**] that there is no such operator of order up to 8 in either direction with polynomial coefficients of total degree at most 6. Increasing the bounds on order and degree further might, of course, help, but this is beyond our current computing capabilities. (For the above assertion, a dense linear system with several thousand variables and equations had to be solved exactly.)

3. A Takayama-Style Approach

By making an ansatz, we could not find a quasi-holonomic annihilating operator, but we could find (and verify) plenty of other operators, $R_1, R_2, R_3, \ldots$ that were not of the quasi-holonomic type. Once it has been verified that these R_i are indeed annihilating operators, we may of course freely choose any operators $P_1, P_2, P_3, \ldots$, and the combined operator

$$P_1 R_1 + P_2 R_2 + P_3 R_3 + \cdots$$

will again be an annihilating operator. In other words, all annihilating operators form a left ideal in the corresponding algebra. Our next step is to find a quasiholonomic combination of the operators $R_1, R_2, R_3, \ldots$ that were found (and verified) by the method of the previous section.

3.1. **Takayama's Algorithm.** Assume we want to find a recurrence for the sum

$$\sum_k f(k,n).$$

In his "holonomic systems approach" [?] the third author proposes to search for an annihilating operator R of $f(n,k)$ of the form

$$R(n,S_k,S_n) = P(n,S_n) + (S_k-1)Q(n,S_k,S_n).$$

Summing over k shows (in case of natural boundaries which we will assume in the following) that $P(n,S_n)$ annihilates the sum. Starting with the annihilator of the summand $f(n,k)$ in $A=\mathbb{Q}(k,n)[S_k,S_n]$, i.e. $\operatorname{Ann}_A f \subseteq A$, one computes an $R(n,S_k,S_n)\in \operatorname{Ann}_A f$ free of k (e.g., by elimination via Gröbner bases). Any such R can be brought to the desired form as above.

Almkvist [?] observed that in the above setting the constraint for Q can be released: The whole proof would go through in the same way if additionally Q depends on k. This fact is exploited in Takayama's algorithm [?, ?] which originally was formulated only in the context of the Weyl algebra. Chyzak and Salvy [?] extended the algorithm to more general Ore algebras (which include also the shift case that we are dealing with) and proposed some optimizations. The idea in short is the following: While in the algorithm above, first k was eliminated and then the part $(S_k-1)Q$ was removed (which corresponds to divide out the right ideal $(S_k-1)A$), the order is now reversed. In Takayama's algorithm we first reduce modulo $(S_k-1)A$ and then perform the elimination of k. The algorithm usually leads to shorter recurrences since we allow more freedom for Q. Second, the elimination is in general much faster since we got rid of Q from the very beginning. Note that Q is not computed at all so we have to assure natural boundaries a priori.

There is one technical complication in this approach. The fact that we are computing in a non-commutative algebra restricts us in the computations after having divided out the right ideal $(S_k-1)A$. In particular, we are not allowed any more to multiply by k from the left. We can easily convince ourselves that otherwise we would get wrong results: Assume we have written an operator already in the form $P+(S_k-1)Q$. Multiplying it by k and then reducing it by $(S_k-1)A$ leads to $kP-Q$ since we have to rewrite $k(S_k-1)$ as $(S_k-1)(k-1)-1$. Because k does not commute with S_k-1 we get the additional term $-Q$ in the result which we lose if we first remove $(S_k-1)Q$ and then multiply by k.

In order to find a k-free operator one needs an elimination procedure that avoids multiplying by k. Let now $R_1,R_2,\dots\in A$ be the operators which generate $\operatorname{Ann}_A f$, and let $R_1',R_2',\dots\in\mathbb{Q}(k,n)[S_n]$ be the corresponding reductions modulo $(S_k-1)A$. For $i=1,2,\dots$ we can write

$$R_i'(k,n,S_n) = R_{i,0}(n,S_n) + R_{i,1}(n,S_n)k + R_{i,2}(n,S_n)k^2+\dots$$

where $R_{i,j}\in A' := \mathbb{Q}(n)[S_n]$. Elimination of k now amounts to finding a linear combination

$$P_1(n,S_n)\begin{pmatrix}R_{1,0}\\R_{1,1}\\R_{1,2}\\ \vdots\end{pmatrix} + P_2(n,S_n)\begin{pmatrix}R_{2,0}\\R_{2,1}\\R_{2,2}\\ \vdots\end{pmatrix} + \dots = \begin{pmatrix}P(n,S_n)\\0\\0\\ \vdots\end{pmatrix}$$

for some $P_1,P_2,\dots\in A'$. The vector on the right hand side then corresponds to the desired k-free operator. More algebraically speaking the computations take place in an A'-module M that is generated by the above vectors. The elimination is achieved by computing a Gröbner basis of this module. Note that $P\in M$ if and only if there exists a $Q\in A$ such that $P+(S_k-1)Q\in\operatorname{Ann}_A f$. For practical purposes we have to truncate the elements of M to a certain length d. The most natural choice is the highest power k^d that appears in the annihilator of f. For any operator R_i we also add

$$k^j R_i \mod (S_k-1)A, \quad 1\le j\le d-\deg_k R_i$$

to the generators of the module. But now we are no longer guaranteed that for any $P,Q\in A$ with $P+(S_k-1)Q\in\operatorname{Ann}_A f$ the operator P is an element of the truncated module. In the unlucky case that no k-free operator was found, the bound d has to be increased. The algorithm works similar in the case of multiple sums where we want to eliminate several variables $k_1, \dots, k_r$.

3.2. **Proof of Gessel's conjecture.** Now back to Gessel's conjecture: Recall that we were looking for a quasi-holonomic operator

$$R(n, i, j, S_n, S_i, S_j) = P(n, S_n) + iQ_1(n, i, j, S_n, S_i, S_j) + jQ_2(n, i, j, S_n, S_i, S_j)$$

where we are mainly interested in P, because Q_1 and Q_2 anyway vanish when we set i and j to 0. This task is very similar to the setting in the previous section and with slight modifications we can apply Takayama's algorithm to solve it. The only difference is that now i and j play the role of $S_k - 1$, and instead of k, we want to eliminate the operators S_i and S_j. Consequently we have to consider the $\mathbb{Q}(n)[S_n]$-module which is generated by $\{S_i^{e_1} S_j^{e_2} | e_1, e_2 = 0, 1, \dots\}$.

For our concrete application, we started with a set of 16 annihilating operators for $f(n; i, j)$. These operators were found by the ansatz described in section 2.2 and verified as proposed in section 2.3. The maximal degree w.r.t. i as well as the maximal degree w.r.t. j is 4. Some of the operators had degree less than 4 in i or j, hence we had to add their corresponding multiples to the set of annihilating operators (which after this step consisted of 24 elements). Next we performed the substitution $i = 0$ and $j = 0$. Finally, the elimination of S_i and S_j using Gröbner bases took about 30 hours and resulted in an operator $P(n, S_n)$ of order 32 and polynomial coefficients of degree 172 in n.

As already pointed out, Takayama's algorithm does not deliver Q_1 and Q_2. However, in principle the full certificate $R(n, i, j, S_n, S_i, S_j)$ can be computed by doing some book-keeping during the run of the algorithm. But in the case of Gessel's conjecture this extra cost would make our computations not feasible any more. We want to emphasize that nevertheless the proof is completely rigorous.

Manuel Kauers, Research Institute for Symbolic Computation, J. Kepler University Linz, Austria

Christoph Koutschan, Research Institute for Symbolic Computation, J. Kepler University Linz, Austria.

Doron Zeilberger, Mathematics Department, Rutgers University (New Brunswick), Piscataway, NJ, USA.

E-mail address: mkauers@risc.uni-linz.ac.at, ckoutsch@risc.uni-linz.ac.at, zeilberg@math.rutgers.edu