Automatic CounTilings

Doron ZEILBERGER[1]

An "Almost" Automatic Proof of Kasteleyn's Formula

In the very special case of two dimer tiles (1×2 and 2×1), Kasteleyn and Fisher&Temperley ([K][FT], see [AS] for a beautiful recent survey) gave their deservedly celebrated beautiful formula for the weight-enumerator of an $m \times n$ rectangle for *arbitrary* (i.e. symbolic) m and n. To wit, if m and n are both even positive integers, and z and z' the variables for the horizontal and vertical dimers respectively (what we called H and V above), then

$$Z_{m,n}(z,z') = 2^{mn/2} \prod_{r=1}^{m/2} \prod_{s=1}^{n/2} \left[z^2 \cos^2 \frac{r\pi}{m+1} + z'^2 \cos^2 \frac{s\pi}{n+1} \right] \quad , \tag{K}$$

with a similar formula when n is odd. At this time of writing, computers can't prove this formula in general. But for any *specific* m (but general(!) n), it is now a routine verification, thanks to our Maple package `TILINGS`. Of course, in practice, it can only be done for small values of m ($m \leq 10$ on our computer), but with bigger and future computers, one would be able to go further. The proof is a beautiful example of (rigorous!) generalization from finitely many cases, combined with 'general-nonsense' linear algebra *handwaving*, that is nevertheless fully rigorous.

Let's call the right side of (K) $W_{m,n}(z,z')$. We have to prove, for our given m, that $W_{m,n}(z,z') = Z_{m,n}(z,z')$ for *all* n. For a fixed even m it is readily seen that $W_{m,n}(z,z')$ is expressible as a product of $m/2$ different dilations of $U_n(z)$, the Tchebycheff polynomials of the second kind. It is well-known (and easily seen) that $U_n(z)$ satisfies a (homogeneous) *linear recurrence equation with constant (i.e. not depending on n) coefficients* of order 2, and hence do does any of its dilation $U_m(\beta z)$, and the product of $m/2$ of these creatures consequently satisfies a linear recurrence equation with constant coefficients of order $2^{m/2}$. Hence its generating function, in t, is rational function whose denominator has degree $2^{m/2}$ and numerator degree $< 2^{m/2}$. We have to prove that this generating function coincides with the 'real thing', the rational function outputted by procedure `GFtH` of `TILINGS`, that also turns out to have the same degrees. But to prove that two rational functions whose numerator degrees is p and denominator degree is q are identical, we only have to check, by elementary linear algebra, that the first $p+q+2$ terms in their Maclaurin series coincide, and this is a routine check, done in our package `TILINGS` by typing `ProveKasteleyn(m);`, for the general case (with z and z'), and `ProveKasteleyn1(m);`, for the straight-enumeration case ($z = z' = 1$).

By hindsight, Kasteleyn and Fisher&Temperley were extremely lucky, since in the very special case of the two dimer tiles, they were able to use graph theory and Pfaffians. The general case

[1] Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA. `zeilberg at math dot rutgers dot edu` ,
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seems, at present, out of reach. Even the monomer-dimer problem is still wide open, let alone an "explicit" expression for the weight-enumerators for tilings of an $m \times n$ rectangle using other sets of tiles. Our Maple package `TILINGS` is a *research tool* that can automatically discover and rigorously prove results for **specific** m but general n. Let's hope that the human can use it to discover new **ansatzes** by which to conjecture and hopefully prove some "explicit" form for the monomer-dimer and more general sets of tiles, where *both* m and n are **symbolic**. From this, one should be able to extract, at least, an "explicit" expression for the so-called *thermodynamic limit*, or at the very least, determine rigorously the *critical exponents*.

I hope to explore these speculations in a subsequent article.