Spelling Out Kathy's Unit 6

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The **Utility** is

$$U(x) = x^{\alpha}\,(Z-x)^{1-\alpha} \quad .$$

We need to find the **first-derivative** if $U(x)$, called $U'(x)$.

$U(x)$ is a **product** of two simpler functions, x^{α} and $(Z-x)^{1-\alpha}$, so we need the **product rule** from calc1.

Recall that the product rule is:

$$(f(x)g(x))' = f'(x)g(x) \, + \, f(x)g'(x) \quad .$$

Taking $f(x) = x^{\alpha}$ and $g(x) = (Z-x)^{1-\alpha}$, we have

$$U'(x) = (x^{\alpha}\,(Z-x)^{1-\alpha})' = (x^{\alpha})'\,(Z-x)^{1-\alpha} + x^{\alpha}\,((Z-x)^{1-\alpha})' \quad . \tag{1}$$

Recall from calc1 that $(x^n)' = nx^{n-1}$ for **any** n, so

$$(x^{\alpha})' = \alpha x^{\alpha-1} \quad .$$

Also recall from the **chain rule**

$$f(g(x))' = f'(g(x))\,g'(x)$$

So

$$((Z-x)^{1-\alpha})' = (1-\alpha)(Z-x)^{1-\alpha-1}\cdot(Z-x)' = (1-\alpha)(Z-x)^{1-\alpha-1}\cdot(-1) = (1-\alpha)(Z-x)^{-\alpha}\cdot(-1) = -(1-\alpha)(Z-x)^{-\alpha}$$

Going back to (1) we have

$$U'(x) = \alpha x^{\alpha-1}\,(Z-x)^{1-\alpha} \, - \, (1-\alpha)x^{\alpha}\,((Z-x)^{-\alpha} \quad .$$

Note: Kathy had a typo she had '+' rather than the correct '−' in the middle.

Factoring out $x^{\alpha-1}(Z-x)^{-\alpha}$ we get

$$U'(x) = x^{\alpha-1}(Z-x)^{-\alpha}(\alpha(Z-x) \, - \, (1-\alpha)x)) \quad .$$

We have to solve $U'(x) = 0$. Dividing by the front $x^{\alpha-1}(Z-x)^{-\alpha}$ we have

$$\alpha(Z - x) \; - \; (1-\alpha)x \; = \; 0 \quad .$$

Simplifying

$$\alpha Z - \alpha x - x + \alpha x \; = \; 0 \quad .$$

Simplifying more

$$\alpha Z - x \; = \; 0 \quad .$$

So $x = \alpha Z$. This is our **critical point**

The rest is a tedious $U''(x)$.