Proof of an Identity Conjectured by Iossif Polterovitch that Came Up in the Agmon-Kannai Asymptotic Theory of the Heat Kernel

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Dedicated to my Teacher, Yakar Kannai, who taught me PDEs and much more

Yakar Kannai's brilliant student, Iossif Polterovitch, who did beautiful work in extending and elaborating the classical Agmon-Kannai theory of the heat kernel, proved, via that theory, the $a = 3n$ case of the following identity:

$$G(a,n;x) := \sum_{j=0}^{a} \frac{1}{(a-j)!(j+n)!(2j+1)} \sum_{k=-j}^{j} \frac{(-1)^k (x+k)^{2j+2n}}{(j-k)!(j+k)!} = 0 \quad , \quad (a \geq 2n) \quad ,$$

that he discovered empirically, using `Mathematica`. He also conjectured that $G(2n-1,n,x) = -1/(2 \cdot 3^n n!)$. In this note I will prove his conjecture.

Introducing the shift operator $Ef(x) = f(x+1)$, we have that $G(a,n;x)$ equals

$$\sum_{j=0}^{a} \frac{(-1)^j}{(a-j)!(j+n)!(2j+1)} \sum_{k=-j}^{j} \frac{(-1)^{j+k} E^k}{(j-k)!(j+k)!} x^{2j+2n} = \sum_{j=0}^{a} \frac{(-1)^j}{(a-j)!(j+n)!(2j+1)!} (E^{1/2} - E^{-1/2})^{2j} x^{2j+2n}$$

Using $E = e^D$ (Taylor's theorem), and writing $f(D) = ((2 \sinh D/2)/D)^2$ (note that $f(D)$ is an even function ($f(D) = 1 + D^2/12 + O(D^4)$)), we have that this equals

$$\sum_{j=0}^{a} \frac{(-1)^j}{(a-j)!(j+n)!(2j+1)!} f(D)^{2j} D^{2j} x^{2j+2n} = \sum_{j=0}^{a} \frac{(-1)^j}{(a-j)!(j+n)!(2j+1)!} f(D)^j \frac{(2j+2n)!}{(2n)!} \; x^{2n}$$

$$= \frac{1}{a!n!} {}_2F_1(n+1/2, -a; 3/2; f(D)) x^{2n} = \frac{1}{a!n!} {}_2F_1(1-n, 3/2+a; 3/2; f(D))(1-f(D))^{a-n+1} \; x^{2n} \quad ,$$

by Euler's transformation: ${}_2F_1(a,b;c;z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z)$ ([B], p. 2). If $a \geq 2n$, then, since $1 - f(D) = O(D^2)$, this equals 0. If $a = 2n-1$, then, since $f(D) - 1 = D^2/12 + O(D^4)$ this equals

$$\frac{(-1)^n}{(2n-1)!n!} {}_2F_1(1-n, 1/2+2n; 3/2; 1) \frac{(2n)!}{12^n} \quad ,$$

which turns out to be $-1/(2 \cdot 3^n n!)$, by Gauss's evaluation ([B], p. 3). □

REFERENCE

[B] W.N. Bailey, "Generalized Hypergeometric Series", Cambridge University Press, 1935.

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