Proof of an Identity Conjectured by Iossif Polterovitch that
 Came Up in the Agmon-Kannai Asymptotic Theory of the Heat Kernel

Doron ZEILBERGER

Dedicated to my Teacher, Yakar Kannai, who taught me PDEs and much more

Yakar Kannai’s brilliant student, Iossif Polterovitch, who did beautiful work in extending and elaborating the classical Agmon-Kannai theory of the heat kernel, proved, via that theory, the $a = 3n$ case of the following identity:

$$G(a, n; x) := \sum_{j=0}^{a} \frac{1}{(a-j)!(j+n)!(2j+1)} \sum_{k=-j}^{j} \frac{(-1)^k(x+k)^{2j+2n}}{(j-k)!(j+k)!} = 0 \quad (a \geq 2n),$$

that he discovered empirically, using Mathematica. He also conjectured that $G(2n - 1, n, x) = -1/(2 \cdot 3^n n!)$. In this note I will prove his conjecture.

Introducing the shift operator $Ef(x) = f(x+1)$, we have that $G(a, n; x)$ equals

$$\sum_{j=0}^{a} \frac{(-1)^j}{(a-j)!(j+n)!(2j+1)} \sum_{k=-j}^{j} \frac{(-1)^{j+k}E^k}{(j-k)!(j+k)!} x^{2j+2n} = \sum_{j=0}^{a} \frac{(-1)^j}{(a-j)!(j+n)!(2j+1)!} (E^{1/2} - E^{-1/2})^{2j} x^{2j+2n}$$

Using $E = e^D$ (Taylor’s theorem), and writing $f(D) = ((2\sinh D/2)/D)^2$ (note that $f(D)$ is an even function ($f(D) = 1 + D^2/12 + O(D^4)$)), we have that this equals

$$\sum_{j=0}^{a} \frac{(-1)^j}{(a-j)!(j+n)!(2j+1)!(2j+2n)} f(D)^{2j} D^{2j} x^{2j+2n} = \sum_{j=0}^{a} \frac{(-1)^j}{(a-j)!(j+n)!(2j+1)!} f(D)^j \frac{(2j+2n)!}{(2n)!} x^{2n}$$

$$= \frac{1}{a!n!} F_1(n+1/2, -a; 3/2; f(D)) x^{2n} = \frac{1}{a!n!} F_1(1-n, 3/2 + a; 3/2; f(D))(1 - f(D))^{a-n+1} x^{2n}$$

by Euler’s transformation: $\_2F_1(a, b; c; z) = (1 - z)^{-a-b} \_2F_1(c-a, c-b; c; z)$ ([B], p. 2). If $a \geq 2n$, then, since $1 - f(D) = O(D^2)$, this equals 0. If $a = 2n - 1$, then, since $f(D) - 1 = D^2/12 + O(D^4)$ this equals

$$\frac{(-1)^n}{(2n-1)!n!} F_1(1-n, 1/2 + 2n; 3/2; 1)(2n)!}{12^n}$$

which turns out to be $-1/(2 \cdot 3^n n!)$, by Gauss’s evaluation ([B], p. 3).

REFERENCE


---