Solutions to Homework 7 SOSC 30100, Sept. 190, 2013

1. (a) the probability that "3" does **not** come up in four tosses of a fair die is $(1 - \frac{1}{6})^4 = (\frac{5}{6})^4$ (since the tosses are *independent*. Hence the probability that a "3" *does* come up is $1 - (\frac{5}{6})^4 = \frac{671}{1296} = .51774\ldots$

(b) the probability that a 'snake's eye' does **not** come up in twenty-four tosses of a pair of fair dice is $(1 - \frac{1}{36})^24 = (\frac{35}{36})^24$ (since the tosses are *independent*. Hence the probability that a snake's eyes *does* come up is $1 - (\frac{35}{36})^24 = .4914\ldots\ldots$.

I would rather bet on (a).

2. (a) If there is no limit to the number of committees that a person can chair, then for each committee that are 50 (independent) choices of chair. Hence the total number is 50^{10}.

If a person can chair at most one committee, then the number is

$$50 \cdot 49 \cdot 48 \cdots ...41 = \frac{50!}{40!} \quad .$$

(There are 50 possibilities for the chair of the first committee, 49 for the chair of the second committee, etc.).

(b) If you don't care who the chair is, then the number of ways is the multinomial coefficient

$$\frac{50!}{5!^6 10!^2} \quad .$$

If in addition, you have to pick a chair for each committee, then the number of possibilities is

$$\frac{50!}{5!^6 10!^2} \cdot (5^6 \cdot 10^2) = \frac{50!}{4!^6 9!^2} \quad .$$

3.

(a) Using Bayes' law, the probability that starting a negative campaign is indeed a **bad** idea, if the focus group decided that it is

$$\frac{(0.4)(0.8)}{(0.6)(0.3) + (0.4)(0.8)} = \frac{0.32}{0.5} = 0.64 \quad .$$

Hence the probability that it is a **good** idea is $1 - 0.64 = 0.36$.

(b) Now the modified *prior* that starting a negative campaign is a good idea is 0.36 and that it is a bad idea is 0.64. If the second focus group decided that it is a good idea after all, then the probability that it is indeed a good idea is

$$\frac{(0.36)(0.7)}{(0.36)(0.7) + (0.64)(0.2)} = \frac{0.32}{0.5} = 0.6631\ldots \quad .$$

(c) If things are reversed, then the probability that starting a negative campaign, after the first focus group is

$$\frac{(0.6)(0.7)}{(0.6)(0.7)+(0.4)(0.2)} = \frac{0.42}{0.5} = 0.84\ldots \quad ,$$

and after the second focus group it is

$$\frac{(0.84)(0.3)}{(0.84)(0.3)+(0.16)(0.8)} = 0.6631\ldots \quad .$$

You get the same probability.

4. (a)

$$\int_b^\infty \frac{1}{b} exp(-\frac{x}{b})]\, dx = 1 - F(b) = 1-(1-exp(-b/b)) = e^{-1} = \frac{1}{e} \quad .$$

(b)

$$\int_0^{2b} \frac{1}{b} exp(-\frac{x}{b})\, dx = F(2b) = (1-exp(-(2b)/b)) = 1-e^{-2} = 1-\frac{1}{e^2} \quad .$$

5.

(a)

$$Pr(X=1) = \frac{3}{28}+\frac{3}{14}+\frac{1}{28} = \frac{10}{28} = \frac{5}{14} \quad ,$$

$$Pr(X=2) = \frac{9}{28}+\frac{3}{14} = \frac{15}{28} \quad ,$$

$$Pr(X=3) = \frac{3}{28} \quad .$$

$$Pr(Y=0) = \frac{3}{28}+\frac{9}{28}+\frac{3}{28} = \frac{15}{28} \quad ,$$

$$Pr(Y=1) = \frac{3}{14}+\frac{3}{14} = \frac{3}{7} \quad ,$$

$$Pr(Y=2) = \frac{1}{28} \quad .$$

(b) The expectation of a random variable is $\sum_{s\in S} X(s)Pr(X=s)$, so:

$$E(X) = 1\cdot Pr(X=1)+2\cdot Pr(X=2)+3\cdot Pr(X=3) = \frac{10}{28}+2\cdot\frac{15}{28}+3\cdot\frac{3}{28} = \frac{10+30+9}{28} = \frac{49}{28} = \frac{7}{4} \quad .$$

Similarly,

$$E(Y) = 0\cdot Pr(Y=0)+1\cdot Pr(Y=1)+2\cdot Pr(Y=2) = 0\cdot\frac{15}{28}+1\cdot\frac{3}{7}+2\cdot\frac{1}{28} = \frac{1}{2} \quad .$$

(c) The variance of a random variable is $\sum_{s \in S} (X(s) - \mu)^2 Pr(X = s)$, where μ is the expectation.

So, since for $E(X) = \frac{7}{4}$,

$$Var(X) = (1-\frac{7}{4})^2 \cdot Pr(X = 1) + (2-\frac{7}{4})^2 \cdot Pr(X = 2) + (3-\frac{7}{4})^2 \cdot Pr(X = 3) = (1-\frac{7}{4})^2 \cdot \frac{10}{28} + (2-\frac{7}{4})^2 \cdot \frac{15}{28} + (3-\frac{7}{4})^2 \cdot \frac{3}{28}$$

Also, since $E(Y) = \frac{1}{2}$,

$$Var(Y) = (0-\frac{1}{2})^2 \cdot Pr(Y = 0) + (1-\frac{1}{2})^2 \cdot Pr(Y = 1) + (2-\frac{1}{2})^2 \cdot Pr(Y = 2) = \frac{1}{4} \cdot \frac{15}{28} + \frac{1}{4} \cdot \frac{3}{7} + \frac{9}{4} cdot \frac{1}{28} \quad .$$

The Covarianace of two random variables X and Y is the expectation of $(X - E(X))(Y - E(Y))$

So

$$Cov(X, Y) = \frac{3}{28}(1 - \frac{7}{4})(0 - \frac{1}{2}) + \cdots$$

(d) They are not independent, since for example, the probability that $X = 2$ and $Y = 1$, $\frac{3}{14}$ is **not** the product of the $Pr(X = 2) = \frac{5}{28}$ and $Pr(Y = 1) = \frac{3}{17}$.

6.

(a) The *expectation* of a continuous random variable whose probability density function is $f(x)$ is

$$\int_{-\infty}^{\infty} x f(x) \, dx.$$

So, in this case

$$E(X) = \int_{x=0}^{\infty} x e^{-x} = 1 \quad .$$

(By integration by parts, or using Maple).

(b)

$$E(X^2) = \int_{0}^{\infty} x^2 e^{-x} = 2 \, dx.$$

(using Maple).

(c)

$$Var(X) = E(X^2) - E(X)^2 = 2 - 1^2 = 2 - 1 = 1 \quad .$$

(d)

$$E(e^{2X/3}) = \int_0^\infty e^{2x/3} e^{-x} = \int_0^\infty e^{-x/3} = \frac{e^{-x/3}}{-1/3}\Big|_0^\infty = -3e^{-x/3}\Big|_0^\infty = -3(0-1) = 3 \quad .$$

7.

(a) $E(X+Y) = E(X) + E(Y) = -2 + 1 = -1$

(b) $Var(X+Y) = Var(X)+Var(Y) = 9+16 = 25$, $Var(X-Y) = Var(X)+Var(Y) = 9+16 = 25$

(c) and (d) too advanced!