Solutions to Homework 2 SOSC 30100, Sept. 9, 2013

1. Here $f(x) = 5x^2 - 4 * x$ the **difference quotient** in general is

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} \quad .$$

In this problem it is:

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{[5(x + \Delta x)^2 - 4(x + \Delta x)] - [5x^2 - 4x]}{\Delta x} \quad .$$

Using the famous $(a + b)^2 = a^2 + 2ab + b^2$ this equals

$$\frac{5x^2 + 10(\Delta x)x + 5(\Delta x)^2 - 4x - 4(\Delta x) - 5x^2 + 4x}{\Delta x} \quad .$$

Simplifying, this equals:

$$\frac{10(\Delta x)x + 5(\Delta x)^2 - 4(\Delta x)}{\Delta x} \quad .$$

Factoring the top, this equals:

$$\frac{(\Delta x)(10x + 5(\Delta x) - 4)}{\Delta x} \quad .$$

Cancelling out Δx from top and bottom gives that this equals

$$10x - 4 + 5\Delta x \quad .$$

Ans. to 1a: The difference quotient is $2x - 4 + 5\Delta x$.

(b) The derivative is the **limit** as $\Delta x \to 0$.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad . = \lim_{\Delta x \to 0} 10x - 4 + 5\Delta x = 10x - 4 + 5 \cdot 0 = 10x - 4 \quad .$$

Ans. to 1b: The derivative dy/dx is $10x - 4$.

Sol. to 1c: Since $f'(x) = 10x - 4$, we have $f'(2) = 10 \cdot 2 - 4 = 20 - 4 = 16$ and $f'(3) = 10 \cdot 3 - 4 = 30 - 4 = 26$ and

2 Using the famous formula, $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ we can simplify q

$$q = \frac{(v + 2)^3 - 8}{v} = \frac{v^3 + 3v^2(2) + 3v(2^2) + 2^3 - 8}{v} = \frac{v^3 + 6v^2 + 12v}{v} = \frac{(v^2 + 6v + 12)v}{v} = v^2 + 6v + 12 \quad .$$

This is a nice polynomial in v so to find the limit at a point, just **plug-it in**

a. $\lim_{v \to 0} q = 0^2 + 6 \cdot 0 + 12 = 12$

b. $\lim_{v\to 2} q = 2^2 + 6\cdot 2 + 12 = 4 + 12 + 12 = 28$

c. $\lim_{v\to a} q = a^2 + 6\cdot a + 12 = 4 + 12 + 12 = 28$

3. We use the constant multiple rule (you can always take a constant out of the differentiation) and the famous rule

$$\frac{d}{dx}x^n = nx^{n-1}$$

a.

$$\frac{d}{dx}\left(-x^{-4}\right) = -\left(\frac{d}{dx}x^{-4}\right) = -(-4)x^{-5} = 4x^{-5} \quad .$$

b.

$$\frac{d}{du}(au^b) = a(\frac{d}{du}u^b = ab\,u^{b-1} \quad .$$

c.

$$\frac{d}{dx}cx^2 = c(\frac{d}{dx}x^2 = c(2x) = 2c\,x \quad .$$

d.

$$\frac{d}{dx}(7x^{1/3}) = 7(\frac{d}{dx}x^{1/3}) = 7\cdot(1/3)x^{-2/3} = \frac{7}{3}x^{-\frac{2}{3}} \quad .$$

4. The product rule is $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.

a. $((9x^2-1)(3x+1))' = (9x^2-1)'(3x+1) + (9x^2-1)(3x+1)' = (18x)(3x+1) + (9x^2-1)3 = 54x^2 + 18x + 27x^2 - 3 = 81x^2 + 18x - 3$

b. $((ax-b)(cx^2))' = (ax-b)'(cx^2) + (ax-b)(cx^2)' = (a)(cx^2) + (ax-b)(2cx) = ac\,x^2 + 2ac\,x^2 - 2bc\,x = 3ac\,x^2 - 2bc\,x$

c.

$$[(2-3x)(1+x)(x+2)]' = (2-3x)'(1+x)(x+2) + (2-3x)(1+x)'(x+2) + (2-3x)(1+x)(x+2)' = (-3)(1+x)(x+2) + (2-3x)(1)($$

$$= -3(1+x)(x+2) + (2-3x)(x+2) + (2-3x)(1+x) \quad .$$

d.

$$[(x^2+3)x^{-1}]' = (x^2+3)'(x^{-1}) + (x^2+3)(x^{-1})' = (2x)(x^{-1}) + (x^2+3)(-1)x^{-2} = 2-(1+3x^{-2}) = 1-3x^{-2} \quad .$$

5. For (a) and (b) it is better to first **simplify**, and then differentiate. For (c) and (d) we need the **quotient rule**.

a.

$$((x^2+3)/x)' = (x+3/x)' = (x+3x^{-1})' = x' + 3(x^{-1})' = 1 + 3(-1)x^{-2} = 1 - \frac{3}{x^2} \quad .$$

b.

$$((x+7)/x)' = (1+7/x)' = (1+7x^{-1})' = 1' + 7(x^{-1})' = 0 + 7(-1)x^{-2} = -\frac{7}{x^2} \quad .$$

The **quotient rule** is:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \quad .$$

c.

$$\left(\frac{4x}{x+5}\right)' = \frac{(4x)'(x+5) - 4x(x+5)'}{(x+5)^2} = \frac{4(x+5) - 4x(1)}{(x+5)^2} = \frac{4x+20-4x}{(x+5)^2} = \frac{20}{(x+5)^2} \quad .$$

d.

$$\left(\frac{ax^2+b}{cx+d}\right)' = \frac{(ax^2+b)'(cx+d) - (ax^2+b)(cx+d)'}{(cx+d)^2} = \frac{(2ax)(cx+d) - (ax^2+b)(c)}{(cx+d)^2}$$

$$= \frac{2ac\,x^2 + 2ad\,x - ac\,x^2 - bc}{(cx+d)^2} = \frac{ac\,x^2 \; + \; 2ad\,x \; - \; bc}{(cx+d)^2} \quad .$$

6. The **chain rule** says

$$\frac{dw}{dx} = \frac{dw}{dy} \cdot \frac{dy}{dx} \quad .$$

Here

$$\frac{dw}{dy} = 2ay \quad , \quad \frac{dy}{dx} = 2bx \quad ,$$

Hence

$$\frac{dw}{dx} = (2ay)(2bx) = 4abxy \quad .$$

Finally we replace y by what it stands for in terms of x, getting

$$\frac{dw}{dx} = 4abxy = 2abx(bx^2+cx) = 2abx^2(bx+c) \quad .$$

7.

$$((16x+3)^{-2})' = (-2)((16x+3)^{-3}) \cdot (16x+3)' = (-2)((16x+3)^{-3}) \cdot (16) = -32(16x+3)^{-3} \quad .$$

By the quotient rule:

$$(\frac{1}{(16x+3)^2})' = \frac{1'(16x+3)^2 - 1((16x+3)^2)'}{(16x+3)^4} = \frac{0 \cdot (16x+3)^2 - 1((2)16x+3))(16)}{(16x+3)^4} = \frac{-32}{(16x+3)^3} = -32(16x+3)^{-3} \quad .$$

The answers are identical because God (or Newton) said so!

8.

a. When x is **VERY BIG** $-7x+5$ and -1 are **insignifcant** compared to $2x^2$ and x respectively, hence it is possible to only retain the leading powers at the top and bottom.

$$\lim_{x\to\infty} \frac{x-1}{2x^2-7x+5} = \lim_{x\to\infty} \frac{x}{2x^2} = \lim_{x\to\infty} \frac{1}{2x} = \frac{1}{2\infty} = 0 \quad .$$

Ans. to 8a: 0.

. Since the top and bottom happen to be 0 at $x = 1$, we can use L'Hôpital's rule:

$$\lim_{x\to 1} \frac{x-1}{2x^2-7x+5} = \lim_{x\to 1} \frac{(x-1)'}{(2x^2-7x+5)'} = \lim_{x\to 1} \frac{1}{4x-7} = \frac{1}{4\cdot 1-7} = -\frac{1}{3} \quad .$$

Ans. to 8b: $-\frac{1}{3}$.

Comment: We can also do it only using algebra

$$\lim_{x\to 1} \frac{x-1}{2x^2-7x+5} = \lim_{x\to 1} \frac{x-1}{(x-1)(2x-5)} = \lim_{x\to 1} \frac{1}{2x-5} = \frac{1}{2\cdot 1-5} = -\frac{1}{3} \quad .$$

9.

First way: Using algebra, $f(x) = x + 1$ when $x \neq 1$. Since $f(1) = 2$ the function given is a coplicated way of writing down $f(x) = x + 1$ that is well-known to be continuous.

Second Way: $f(x)$ is continuous when $x \neq 1$ (since it is a quotient of two continuous functions, and the denominator does not vanish). The limit of $f(x)$ as x goes to 1 happens to be 2 (either by L'Hôpital or using algebra). Since the limit of $f(x)$ as x goes to 1 equals $f(1)$, it is also continuous at $x = 1$, hence it is continuous over all the real line.

10. By the product and chain rules:

$$f'(x) = (\frac{1}{2}(x^2+4x-9)^3x^{-2})' = \frac{1}{2}((x^2+4x-9)^3x^{-2})' =$$

$$\frac{1}{2}[((x^2+4x-9)^3)'x^{-2}+((x^2+4x-9)^3)(x^{-2})'] = \frac{1}{2}[(3(x^2+4x-9)^2(2x+4)x^{-2}+((x^2+4x-9)^3)(-2)x^{-3}]quad.$$

I hope that you don not expect me to simplify!