A COMBINATORIAL PROBLEM THAT AROSE IN BIOPHYSICS

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The purpose of this note is to prove the following result that was conjectured by T.L.Hill ([1], [2], p.148) in the course of his investigations of the "surface" properties of some long multi-stranded polymers.

THEOREM: Let s be a positive integer, and for any non-negative integer m, let $R(m)$ be the number of solutions, in *integers*, $(m_1, ...m_s)$, of the system

$$m_1 + ... + m_s = 0, \tag{1a}$$

$$|m_1| + ... + |m_s| = 2m. \tag{1b}$$

Then,

$$Q(\rho) := \sum_{m=0}^{\infty} R(m)\rho^m = (1-\rho)^{-(s-1)} \sum_{k=0}^{s-1} \binom{s-1}{k}^2 \rho^k.$$

PROOF: It is readily seen that R(m) is the coefficient of $\rho^m t^0$ in

$$\left(\sum_{k=-\infty}^{\infty} t^k \rho^{|k|/2} \right)^s = [\rho^{1/2}t^{-1}/(1-\rho^{1/2}t^{-1}) + 1 + (\rho^{1/2}t/(1-\rho^{1/2}t)]^s =$$

$$(1-\rho)^s (1-\rho^{1/2}t)^{-s} (1-\rho^{1/2}t^{-1})^{-s} \tag{2}$$

Thus, $Q(\rho)$ is the coefficient of t^0 in the right side of (2). Expanding the last two terms in the right side of (2) by Newton's binomial formula, and collecting the coefficient of t^0 we get

$$Q(\rho) = (1-\rho)^s \sum_{k=0}^{\infty} \binom{s+k-1}{s-1}^2 \rho^k \tag{3}$$

Recall that the hypergeometric function F(a,b;c;z) is defined by (e.g. [3] chapter 4):

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \tag{4}$$

where $(a)_n := a(a+1)...(a+n-1)$.

One of the most celebrated formulas in the theory of hypergeometric functions is Euler's transformation ([3], theorem 21, p.60):

$$F(a,b;c;z) = (1-z)^{c-a-b} F(c-a, c-b; c; z). \tag{5}$$

Converting the right side of (2) into hypergeometric notation, using (5) , and converting back yields:

$$Q(\rho) = (1-\rho)^s F(s,s;1;\rho) = (1-\rho)^s (1-\rho)^{1-2s} F(1-s, 1-s; 1; \rho) = (1-\rho)^{-(s-1)} \sum_{k=0}^{s-1} \binom{s-1}{k}^2 \rho^k$$

Q.E.D.

The same method of proof can be applied to treat the more general problem where the 0 at the left side of (1a) is replaced by a general integer i.

REFERENCES

1. Hill,T.L.,*Effect of fluctuating surface structure and free energy on the growth of linear tubular aggregates***49**, 1017-1031 (1986).

2. Hill,T.L.,"Linear Aggregation Theory in Cell Biology", Springer-Verlag, New York, (1987).

3. Rainville,Earl D., "Special Functions", Chelsea, Bronx, New York, (1971).