Dave ROBBINS's ART of GUESSING

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Abstract: *I will try to guess how Dave Robbins guesses such beautiful mathematical results.*

Dave Robbins's 22 items in **Mathscinet** are but the tip of an iceberg. At least ninety percent of his work is classified, and we will have to wait fifty years to enjoy it. Most people are unaware how much we owe to such classified work. Everybody knows who Hellman and Diffie, and Rivest, Shamir, and Adelman are, but how many people ever heard of their shadow analogs, Malcolm Williamson, Cliff Cocks, and James Ellis? How many people heard of Oscar Rothaus, who recently passed away? Yet, his brainchild, *Hidden Markov Models*, is a household name in speech recognition, bioinformatics and elsewhere.

Luckily, the unclassified ten percent of Dave's work is more than enough to make him immortal. Besides, Dave is one of the greatest problem posers and problem solvers I know, and I am sure that the classified work presents to him lots of challenges that he so excels in.

Our story starts a long long time ago, with **Patriarch Abraham**, who, at least according to tradition, authored *Sefer Yetsira*, "The book of Creation", a Cabalistic text that was compiled about 1700 years ago. The Cabala is a Combinatorial 'Theory of Everything', both spiritual and physical, and anagrams (of Hebrew letters), called *temurot* are of fundamental importance.

There you can find the following assertions.

"*two stones make two houses*,
three stones make six houses ,
four stones make four and twenty houses,
five stones make a hundred and twenty houses,
six stones make twenty and seven hundred houses,
seven stones make forty and five thousand houses",
and it concludes:
"*from then on, you get what the mouth can't utter and the ear can't hear*".

This was generalized many years later by Rabbi Levi Ben Gerson, who in his 1321 'Book of Number' stated and *proved* a very general theorem.

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First version: April 23, 2004 . Transcript of an invited talk given on June 30, 2003, 9:30-10:22 am, at the Conference celebrating Dave Robbins (Aug. 12, 1942-Sept. 4, 2003), held (June 29-30, 2003) at the Center for Communication Research of the Institute for Defense Analyses, Princeton, NJ. Accompanying Maple programs, as well as input and output files discussed in this article, can be viewed and downloaded from `http://www.math.rutgers.edu/~zeilberg/DaveRobbins/guess.html`. Supported in part by the NSF.

Theorem: The number of houses one can build from N stones equals one times two times ... times N.

Levi's proof is one page long, and would meet the standards of rigor of the strictest contemporary referee.

Levi Ben Gerson, whom I doubt any of you ever heard of, was one of the greatest mathematicians and astronomers of the late middle ages, and was also a great expositor. No one in this audience could disagree with Levi's following quote:

"*In order to be a good computer it is necessary to Understand the methods of calculations.*"

Of course, by 'computer' he meant a *human* computer. In order to achieve this pedagogical goal, Levi Ben Gerson divided his book into a theoretical part, complete with rigorous proofs à la Euclid, and an algorithmic part, with detailed examples.

Five hundred and forty five years later, another cleric, *Reverend Charles Lutwidge Dodgson* wrote (*Proc. Royal Soc.* **84** (1866), 150-155) an article entitled *Condensation of Determinants* and subtitled *Being a new and brief method for computing their arithmetical values.*

Here is an example, taken from Dodgson's paper:

$$\begin{pmatrix} -2 & -1 & -1 & -4 \\ -1 & -2 & -1 & -6 \\ -1 & -1 & 2 & 4 \\ 2 & 1 & -3 & -8 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & -1 & 2 \\ -1 & -5 & 8 \\ 1 & 1 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 8 & -2 \\ -4 & 6 \end{pmatrix} \rightarrow -8.$$

Each iteration consists of forming a new matrix by taking consecutive 2 by 2 minors of the current matrix, and dividing by the corresponding entry of the previous matrix (where the 0^{th} matrix consists of all 1's).

The obvious drawback of this method, as a numerical procedure, is that, as Dodgson put it, "*The process cannot be continued when ciphers occur*". To get around this, our dear Reverend 'cheats' and uses row and/or column operations, in other words, Gaussian elimination, that by the way has the same computational complexity. The advantage of Dodgson's method is that all divisions are exact, so if the entries of our matrix are integers, all intermediate results will be integers as well.

Dodgson's method can be phrased, symbolically, as follows.

Initial Conditions:

$$A^{(0)}_{i,j} = 1 \quad ; \quad A^{(1)}_{i,j} = a_{ij} \quad , \quad 1 \leq i,j \leq n \quad .$$

Iteration Step: For $2 \leq k \leq n$, $1 \leq i,j \leq n-k+1$,

$$A^{(k)}_{ij} = \frac{A^{(k-1)}_{ij} A^{(k-1)}_{i+1,j+1} - A^{(k-1)}_{i,j+1} A^{(k-1)}_{i+1,j}}{A^{(k-2)}_{i+1,j+1}} \quad . \qquad (CLD)$$

Final Output: $\det A = A_{1,1}^{(n)}$.

If $A = (a_{ij})$ is a *symbolic* (generic) matrix, then the final output would be the symbolic determinant:

$$\det A = \sum_{\pi \in S_n} (sgn\, \pi) \prod_{i=1}^{n} a_{i,\pi(i)} \quad ,$$

which is a sum parameterized by permutations. Of course using this formula is a terrible way to actually compute numerical determinants.

About 115 years later (ca. 1980), another genius, Dave Robbins, in collaboration with Howard Rumsey, had a brilliant idea.

Replace the $-\mathbf{1}$ by **lambda** and let

$$A_{ij}^{(k)} = \frac{A_{ij}^{(k-1)} A_{i+1,j+1}^{(k-1)} + \lambda A_{i,j+1}^{(k-1)} A_{i+1,j}^{(k-1)}}{A_{i+1,j+1}^{(k-2)}} \quad , \qquad (\lambda - CLD)$$

and *define* the λ-determinant, $\det_\lambda A$, to be equal to the final output, $A_{1,1}^{(n)}$. Surprisingly, this turns out to be a (Laurent) polynomial in the entries!

Using the now long-defunct computer-algebra program **Altran**, Dave Robbins and Howard Rumsey first conjectured, and then *proved*, the amazing generalization of the expanded form of the symbolic determinant:

$$det_\lambda(A) = \sum_{B \in ASM(n)} \lambda^{I(B)} (1 + \lambda^{-1})^{N(B)} \prod_{1 \le i,j \le n} a_{ij}^{B_{ij}} \quad , \qquad (DaveHoward)$$

where $ASM(n)$ is the set of now famous *alternating sign matrices*, to be described shortly, $I(B)$ is the *number of inversions* and $N(B)$ is the number of $(-1)s$. Note that when $\lambda = -1$, only those ASMs with $N(B) = 0$ contribute to the sum, and one gets back the formula for the traditional determinant.

The *discovery* of this natural object was a beautiful example of what today is called *Experimental Mathematics*, that is becoming increasingly fashionable. But Dave was an 'Experimental Mathematician when Experimental Math was not yet cool'. Also his tools, about 25 years ago, were much more primitive. With the benefit of hindsight, and much more powerful software and hardware, it is fun to 're-enact' in Maple (for example), his beautiful discovery, *ab initio.*

Here are the few lines of Maple code needed:

```
Tg:=proc(A,B,n,g) local i,j:
[seq([seq(normal(expand((B[i][j]*B[i+1,j+1]+
g*B[i+1][j]*B[i][j+1])/A[i+1][j+1])),j=1..n-1)],i=1..n-1)]:  end:

det1g:=proc(B,g) local A,n,i,B1,B1o:
n:=nops(B): A:=[[1$n]$n]:  B1:=B: for i from 1 to n-1 do
```

```
B1o:=B1:  B1:= expand(Tg(A,B1,n-i+1,g)):  A:=B1o:  od:
expand(B1[1][1]):end:

Detg:=proc(n,b,g) local i,j:  expand(det1g([seq([seq(b[i,j],i=1..n)],j=1..n)],g)):end:

Dave:=proc(t,b,n) local i,j:  [seq([seq(degree(t,b[i,j]),j=1..n)],i=1..n)]:end:

MRR:=proc(n) local b,g,A,i:  A:=Detg(n,b,g):  seq(Dave(op(i,A),b,n),i=1..nops(A)):end:

MRRp:=proc(n) local A,i:  with(linalg):A:=MRR(n):
seq(print(matrix(n,n,A[i])),i=1..nops(A)):end:
```

Procedure `Tg` is an implementation of (CLD) with $A = A^{(k-2)}$ and $B = A^{(k-1)}$, outputting $A^{(k)}$.

Procedure `det1g` iterates `Tg` $n-1$ times, thereby computing the $\lambda-$ determinant of an arbitrary (square) matrix B. I represent matrices in Maple as lists of lists, to preserve their combinatorial nature, in order not to get contaminated with irrelevant linear-algebra data structures.

Procedure `Detg` applies Dodgson's rule (i.e. Procedure `det1g`) to the *generic* $n \times n$ matrix $(b[i,j])_{1 \le i,j \le n}$, where the $b[i,j]$ are *symbolic*, i.e. *commuting indeterminates.*

For any specific integer n, $Detg(n,b,g)$ is a huge *Laurent polynomial* in the n^2+1 variables $b_{i,j}$ and g. Let's forget about the g for now, and consider it as a Laurent polynomial in the $b_{i,j}$ with coefficients that are polynomials in g. But a *poly*-nomial is a linear combination of *mono*-mials, and a natural question is: which of the monomials show up? Given a monomial, we can look at the exponents of the variables $b_{i,j}$, and get a certain matrix. Hence characterizing the monomials

$$\prod_{1 \le i,j \le n} b_{i,j}^{B_{i,j}}$$

is equivalent to characterizing the integer-matrices $(B_{i,j})_{1 \le i,j \le n}$. Procedure `Dave` is simply the translation from monomials to exponent-matrices.

Finally, Procedure `MRR` lists all the exponent-matrices of the monomials that show up in `Detg(n,b,g)`, and `MRRp(n)` prints them out 'nicely'.

Once I finished writing the above Maple code , my computer was ready to re-enact the historic discovery of Alternating Sign Matrices by Dave Robbins, that he did about twenty-five years ago.

For example, typing `MRRp(3)` yields:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad .$$

Perhaps the case $n = 3$ is not enough to see what is going on, but you don't have to be a Dave Robbins to be able, after glancing at the output of "`MRRp(5);`"
(see `http://www.math.rutgers.edu/ zeilberg/DaveRobbins/oMRR5`), to notice that these matrices have

(i) entries that are taken from $\{-1, 0, 1\}$,
(ii) row- and column- sums equal to 1,
(iii) in each row and column, the non-zero entries alternate in sign,

et voilà, we have re-discovered Dave Robbins's immortal object, that he named *Alternating Sign Matrices* (henceforth ASM).

Once we have the notion of ASM, it is not hard, with Maple once again, to conjecture ($DaveHoward$). Just find the coeff. corresponding to each of the monomials, and `factor` it, and realize that the coeff. of the monomial that corresponds to $B_{i,j}$ is $g^{I(B)}(1 + 1/g)^{N(B)}$, where $I(B)$ is the *number of inversions* (appropriately defined) and $N(B)$, even more simply, is the number of -1's in $(B_{i,j})$.

An $n \times n$ ASM with no -1's is simply a *permutation-matrix* of order n, and their number, thanks to Levi Ben Gerson, equals $n!$. The next natural question that Dave and his collaborators asked was: How many $n \times n$ ASM's are there? The Mills-Robbins-Rumsey conjecture was yet another tour-de-force in *experimental mathematics*, long before that term existed.

Before anyone, even Dave, can conjecture an explicit expression for an enumerating sequence, we need at least ten terms (unless the sequence is really trivial). How can we crank out, say, $\{|ASM(n)|\}_{n=1}^{20}$? Definitely **not** by typing

```
seq(nops(MRR(n)),n=1..20)    .
```

This will explode by $n = 8$ (even nowadays), because of the super-exponential growth of the enumerating sequence, that by the way, looks like

$$\begin{array}{c}
1\\
2\\
7\\
42\\
429\\
7436\\
218348\\
10850216\\
911835460\\
129534272700\\
31095744852375\\
12611311859677500\\
8639383518297652500\\
9995541355448167482000\\
1952907623466127710489720 0\\
644271857034256893568967438 40\\
3588692019161376014474861564172 96\\
33748606392587505622695144915229254 56\\
535803508339843488888786461497090923132 44\\
1436038934715538200913155682637051204376827212\\
6497129499980842789584790438052414353885855143775 7
\end{array} .$$

Note the parabolic shape indicating that the rate of growth is $10^{O(n^2)}$.

How did Dave et. al. actually find the first 20 or so members of the enumerating sequence for ASMs, that enabled them to make their beautiful conjecture? First they needed a more **enumeration-friendly data-structure**. Starting with an ASM, Dave first formed the partial column-sums (for each and every column) yielding, e.g.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} .$$

Because of the *alternating* condition, the new matrix has all its entires 0's and 1's, and because every column starts and ends with a 1, the bottom row (that gives the respective column-sums), is the 'all-ones' row. The next thing to do is to *compactify* the information by recording, for each row, the location of the 1's. In this example we get

$$\begin{array}{ccccccccc}
 & & & & 3 & & & & \\
 & & & 3 & & 4 & & & \\
 & & 1 & & 3 & & 5 & & \\
 & 1 & & 2 & & 4 & & 5 & \\
1 & & 2 & & 3 & & 4 & & 5
\end{array} .$$

Dave called these triangles **monotone triangles**. The ASM conditions are easily seen to translate to the conditions that the entries in each row are **strictly** increasing, and every entry is weakly between the two entries right below it. If you relax the condition that the the entires in every row are *strictly* increasing, and only insist on *weak*-increase, then these 'new' creatures are classical: the so-called **Gelfand-Zeitlin** patterns, that, in turn, are equivalent to even more classical algebraic-combinatorial objects: Young-tableaux! Using classical stuff about Schur functions, one easily gets that the number of Gelfand-Zeitlin patterns with bottom row equaling $12\dots n$ is

$$2^{n(n-1)/2} \quad .$$

Is there a 'nice' formula for the **strict** analog? Well, let's continue in our efforts to compile a table of the first 20 entries, and perhaps we'll see some 'pattern' in the enumerating sequence for these patterns.

As almost always in enumeration (and elsewhere), one needs to consider a more *general* problem, that would hopefully enable a **recurrence scheme**.

So let's not be so narrow-minded and only try to find the number of monotone-triangles whose bottom row is $12\dots(n-1)n$, and instead define $F(a_1,\dots,a_n)$ to be the number of monotone triangles whose bottom row is $a_1\dots a_n$.

Now once the bottom row is fixed at $a_1\dots a_n$, what can the second-row from the bottom be? Let it be $b_1\dots b_{n-1}$, then, clearly:

$$a_1 \le b_1 \le a_2 \le b_2 \le \dots \le b_{n-1} \le a_n \qquad (Cond1)$$

and

$$b_1 < b_2 < \dots < b_{n-1} \quad . \qquad (Cond2)$$

and

$$F(a_1,\dots,a_n) = \sum_{(b_1,\dots,b_{n-1})} F(b_1,\dots,b_{n-1}) \quad , \qquad (Recurrence)$$

where the summation is over all the b's that satisfy $(Cond1)$ and $(Cond2)$.

Now, of course, $F(a_1,\dots,a_n)$ was only a *stepping-stone* for computing $|ASM(n)|$, and

$$|ASM(n)| = F(1,2,\dots,n) \quad .$$

Equipped with $(Recurrence)$, it was not too hard to compute $|ASM(n)|$ for $n \le 20$ even way back in 1980.

The next thing Dave did was to factorize these integers, and lo and behold, they seemed to be 'round', i.e. their prime factors are small. Assuming that you have already implemented the above procedure to enumerate ASMs, and called it `a(n)` , then typing

```
seq(ifactor(a(i)),i=1..7);
```

would yield

$$1, (2), (7), (2)(3)(7), (3)(11)(13), (2)^2(11)(13)^2, (2)^2(13)^2(17)(19) \quad .$$

So there must be something going on here. But how did Dave *conjecture* the right formula? Well, 'roundness' usually means that there is an expression featuring factorials. The claim to fame of the factorial function $f(n) := n!$ is that the *ratio* $f(n+1)/f(n)$ equals n, and in general, any product and/or quotient of factorials of affine-linear expressions has the property that $f(n+1)/f(n)$ is a *rational function* of n.

So let's investigate the ratios

$$b(n) := \frac{a(n+1)}{a(n)} \quad ,$$

but, alas, they still do not seem to be given by a rational function. But why not *do it again?*. Let's take the 'ratios of ratios', and investigate

$$c(n) := \frac{b(n+1)}{b(n)} = \frac{a(n)a(n+2)}{a(n+1)^2} \quad ,$$

and try the *ansatz* of rational functions. After trying, in vain, rational functions of degree 1, we are naturally lead to try

$$c(n) = \frac{\alpha n^2 + \beta n + \gamma}{\alpha' n^2 + \beta' n + \gamma'} \quad , \qquad (Ansatz)$$

for some, as yet, *undetermined* coefficients $\{\alpha, \beta, \gamma, \alpha', \beta', \gamma'\}$. Now all we need is six equations to determine these six unknowns. But, just like in curve-fitting in experimental science, or sequence-guessings in IQ tests, we need some over-determination, so let's plug in $(Ansatz)$, $n = 0, 1, 2, , 3, 4, 5, 6$ getting the system of equations (I am reproducting them here for pedagogical reasons, of course, in real life it is all done automatically and internally on the computer):

$$\frac{1 \cdot 7}{2^2} = \frac{\alpha \cdot 1^2 + \beta \cdot 1 + \gamma}{\alpha' \cdot 1^2 + \beta' \cdot 1 + \gamma'} \quad , \quad \frac{2 \cdot 42}{7^2} = \frac{\alpha \cdot 2^2 + \beta \cdot 2 + \gamma}{\alpha' \cdot 2^2 + \beta' \cdot 2 + \gamma'} \quad , \quad \frac{7 \cdot 429}{42^2} = \frac{\alpha \cdot 3^2 + \beta \cdot 3 + \gamma}{\alpha' \cdot 3^2 + \beta' \cdot 3 + \gamma'} \quad ,$$

$$\frac{42 \cdot 7436}{429^2} = \frac{\alpha \cdot 4^2 + \beta \cdot 4 + \gamma}{\alpha' \cdot 4^2 + \beta' \cdot 4 + \gamma'} \quad , \quad \frac{429 \cdot 218348}{7436^2} = \frac{\alpha \cdot 5^2 + \beta \cdot 5 + \gamma}{\alpha' \cdot 5^2 + \beta' \cdot 5 + \gamma'} \quad ,$$

$$\frac{7436 \cdot 10850216}{218348^2} = \frac{\alpha \cdot 6^2 + \beta \cdot 6 + \gamma}{\alpha' \cdot 6^2 + \beta' \cdot 6 + \gamma'} \quad .$$

Clearing denominators, and `solve`-ing the resulting system of linear equations, Maple answers with the *conjecture*

$$c(n) = \frac{3(3n+4)(3n+2)}{4(2n+3)(2n+1)} \quad . \qquad (ConjectureRatRat)$$

To give it more plausibility, we can now test it for the first 20 or whatever values of $c(n)$ that we have.

Unfolding the ratio, we get immediately

$$b(n) = \frac{\binom{3n+1}{2n+1}}{\binom{2n-1}{n-1}} \quad , \qquad (ConjectureRat)$$

that, in turn, implies

$$a(n) = \prod_{j=1}^{n} \frac{\binom{3j-2}{2j-1}}{\binom{2j-3}{j-2}} \quad ,$$

or equivalently:

$$a(n) = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} \quad . \qquad (MRRconjecture)$$

But, if this is not evidence enough for you, Dave et. al. found much more compelling evidence, by discovering a beautiful *refinement* of their amazing conjecture.

It is obvious that the bottom row of any ASM can only have one 1 and no -1's (or else it wouldn't be *alternating*). Let $A(n,k)$ be the number of $n \times n$ alternating sign matrices with the sole 1 of the bottom row residing at the k-th column. Equivalently, $A(n,k)$ is the number of *monotone triangles* whose bottom row is $1, 2, \dots, \hat{k} \dots, n$. In terms of F defined above by (*Recurrence*), we have

$$A(n,k) = F(1, 2, \dots, k-1, k+1, \dots, n) \quad .$$

Now Dave formed the famous *Robbins triangle*, whose rows list $A(n,1), A(n,2), \dots, A(n,n)$ $(n \geq 2)$:

$$1, 1$$

$$2, 3, 2$$

$$7, 14, 14, 7$$

$$42, 105, 135, 105, 42$$

$$429, 1287, 2002, 2002, 1287, 429 \quad .$$

Next Dave took the *ratios* of neighbors in each row, getting

$$1$$

$$2/2$$

$$2/3 \quad 3/2$$

$$2/4 \quad 5/5 \quad 4/2$$

$$2/5 \quad 7/9 \quad 9/7 \quad 5/2 \quad .$$

Let's take a look at the second and fourth row. Why write 2/2 and not 1/1, 2/4 and not 1/2, 5/5 and not 1/1 (and similarly for ratios in rows further down, that I did not display). After all, once you renounce writing fractions in their reduced (canonical) forms, there are *infinitely* many possibilities (why not write 1/2 as 1003/2006, say?). According to Dave, this was 'the only creative part', i.e. deciding *how* to express the fractions so that a *pattern* emerges. According to some human supremacists, it is this gift that distinguishes human-kind from machine-kind. A priori we have a 'search-space' that is infinitely large, but smart humans like Dave Robbins have a knack for cutting this infinite haystack to a very manageable one.

Anyway, once Dave 'reduced' the fractions in the 'right' way, both numerators and denominators formed, amazingly, Pascal-like triangles, with the same rule of formation, only different initial conditions, and it followed immediately that

$$\frac{A(n,k)}{A(n,k+1)} = \frac{\binom{n-2}{k-1} + \binom{n-1}{k-1}}{\binom{n-2}{k-1} + \binom{n-1}{k}} \quad ,$$

which easily simplifies to

$$\frac{A(n,k)}{A(n,k+1)} = \frac{k(2n-k-1)}{(n-k)(n+k-1)} \quad .$$

With the benefit of hindsight, it turns out that Dave's very creative human heuristic guessing was a **red herring**, even a **category mistake**, since the ratios turned out to be *rational functions*, not *quotients of binomials.*

Nowadays, if you suspect that a discrete function of two variables, in our case

$$B(n,k) := \frac{A(n,k)}{A(n,k+1)} \quad ,$$

is a rational function, just use a Maple procedure like my own `GuessRat`, to guess the rational function, by plugging-in specific values, just like we did above for $c(n)$, the 'ratio of ratios' sequence of $a(n)$.

Here are the few lines of Maple code needed to *re-enact* the discovery of the *Refined Alternating Sign Matrix Conjecture.*

```
U:=proc(a) local n,i,j,lu:option remember:
n:=nops(a):  if n=1 then RETURN([]) :  elif n=2 then
RETURN(seq([i],i=a[1]..a[2])) :
else lu:=U([op(1..n-1,a)]):
seq(seq([op(lu[i]),j],j=max(lu[i][n-2]+1,a[n-1])..a[n]), i=1..nops(lu)):
fi:  end:


F:=proc(a) local gu,i:  option remember:
if nops(a)=1 then 1 else gu:=U(a):  add(F(gu[i]),i=1..nops(gu)):fi:end:

An:=proc(n) local i:  F([seq(i,i=1..n)]):end:
```

```
Ank:=proc(n,k) local i:  F([seq(i,i=1..k-1),seq(i,i=k+1..n)]):end:

Bnk:=proc(n,k):  Ank(n,k)/Ank(n,k+1):end:

GuessASM:=proc(n,k,d) local x,y:
subs(y=k, x=n-k, GuessRat((x,y)->Bnk(x+y,y),[x,y],d,1)):  end:
```

Here, procedure `U(a)` recursively finds the set of b's satisfying $(Cond1)$ and $(Cond2)$ above, while procedure `F(a)` implements $(Recurrence)$. `An(n)`, `Ank(n,k)` and `Bnk(n,k)` output $a(n), A(n,k)$ and $B(n,k)$ respectively, and finally `GuessASM` guesses the rational function for $B(n,k)$. It uses procedure `GuessRat`, a guessing program that uses linear algebra to guess the form of a conjectured rational function from sufficient data, that can be downloaded from the webpage of this paper.

After typing the above lines of code, modulo `GuessRat` (that is fairly short), typing

```
GuessASM(n,k,4) ;
```

would immediately return

$$\frac{k(2n-k-1)}{(n-k)(n+k-1)} \quad , \qquad (RefinedRatio)$$

that is equivalent to the statement of the *refined* Alternating Sign Matrix Conjecture that, in turn, implies the original, unrefined, ASM conjecture.

Indeed $(RefinedRatio)$ implies

$$A(n,k) = \frac{\binom{n+k-2}{k-1}\binom{2n-k-1}{n-k}}{\binom{2n-2}{n-1}} \cdot A(n,1) \quad .$$

But, of course, $A(n,1) = a(n-1)$, so we get, thanks to the *Chu-Vandermonde identity* ,

$$a(n) = \sum_{k=1}^{n} A(n,k) = \frac{\binom{3n-2}{2n-1}}{\binom{2n-3}{n-2}} a(n-1) \quad ,$$

that enables one to crank out as many terms of the *conjectured* sequence as desired.

Today, whenever you discover a new **integer sequence** you go to

`http://www.research.att.com/~njas/sequences` ,

that gets updated **daily**, and you can find out right away whether your alleged discovery is indeed new or a *re*-discovery. Yesterday I went there and entered:

`1,1,2,7,42,429`

and immediately got a full description, with lots of references. But what *pleased* me most was the *name*:

Name: Robbins Numbers .

So Neil Sloane, the Guardian of the Integer Sequence Treasure Trove, did the right thing and named $1, 1, 2, 7, 42, 429 \ldots$ after Dave, thereby immortalizing him by inducting him into the class of Fibonacci, Catalan, Lucas, Bell, and other luminaries. This is in spite of the fact that the paper was co-authored with Bill Mills and Howard Rumsey, and that the Robbins Numbers were anticipated by George Andrews. I am sure that Bill and George whole-heartedly agree with Neil's decision (and Howard would have, had he been alive).

Anyway, in the pre-web **bad old days** of the paper tyranny, the *Handbook of Integer Sequences* got updated every 21 years (the first edition was 1974, the second 1995). But even though there was no Sloane website back in 1980, the phone had already been invented, so one could call the *Godfather*, **Richard Stanley**, who, in Dave's words: **startled** them by telling them that Partitions guru **George Andrews** has discovered (or rather invented!) this sequence when he invented **Descending Plane Partitions**.

There is a very old, still unresolved, philosophical problem. Is math **invented** or **discovered**? According to my Patron Saint, James Joseph Sylvester, some is invented and some is discovered. I believe that Andrews's descending plane-partitions belong to the invented part. The definition (that I will spare you!) is amongst the most contrived and ugly in the whole of combinatorics. But how could someone with such good taste as George Andrews even think of inventing a new and artificial kind of partitions, when they are so many natural kinds to keep us busy?

What happened is that George found a powerful new method, that helped him prove the 80-year-old MacMahon conjecture about Symmetric Plane Partitions. Having done so, he was looking for other customers to sell the new method to. Not finding any, he invented this new nail, that was ideally suited to be hammered by his new method.

But it was just as well! So one shouldn't look down at 'generalizations for the sake of generalizations', and 'new combinatorial objects for the sake of being enumerable by an existing method'. Math, like, money, does not smell, and sometimes the most artificial (to human eyes) math can lead, directly or indirectly, to exciting and **natural** new math.

In this case, Mills, Robbins, and Rumsey got **hooked** on plane-partitions, and soon found out about the then open **Macdonald Conjecture**, that q-enumerates the **natural** class of **Cyclically Symmetric Plane Partitions**. George Andrews (who else) had then recently accomplished the 'straight enumeration' (the case $q = 1$), in a very complicated *Inventionae* paper, but the general case remained wide open. This conjecture was stated in Ian Macdonald's classic '*Symmetric Functions*', that just came out then, and in Richard Stanley's book review, in the Bulletin (of the Amer. Math. Soc.), Stanley called it "**The most interesting open problem in all of enumerative combinatorics**".

Dave, being a compulsive (and of course, brilliant) **problem-solver**, taped Stanley's quote on his office door, and assisted by Bill Mills and Howard Rumsey, proved Macdonald's full conjecture in a seminal *Inventionae* paper. It was one of the very few papers on combinatorics published back then by that snooty and fussy periodical.

So MRR never proved their ASM conjecture, but in the process, proved something at least as interesting. This is the beauty of math (and life). It is good to have goals, but it is not so important to achieve them, since in your search for the initial goals you might find even greater treasures!

I must also mention that Descending Plane Partitions are not so artificial after all. Later on, Mills, Robbins and Rumsey found amazing refinements of the fact that the number of descending plane partitions equals the number of ASMs of the same size, by finding a three-parameter refinement. This conjecture is still open, as far as I know.

Postscript: How I Proved the ASM Conjecture

So far, my account was based on hearsay, conversations with Dave, and his published works. But now I can start an **eye-witness** account.

In May 1982 I was just recently converted to combinatorics, and thanks to my late mentor, Joe Gillis, I was invited to participate at the combinatorics Oberwolfach meeting, organized by Dominique Foata. Now if you know Dominique, you would know that he always begs people **not** to talk, and laments that people always insist on giving talks, while it is much better to have less talks and more time for informal discussions and interactions (what Foata calls the 'spirit of the early Oberwolfach', before it got ruined by the establishment). So Foata very reluctantly agrees to have anyone speak for the full 50 minutes.

But Dave's first talk, about the MRR proof of Macdonald's CSPP conjecture, was so good, and it hinted at the intriguing ASMs, that Dominique, (and everyone else!) *begged* Dave to give a second fifty-minute talk, about ASMs and their conjectured enumeration.

So Dave was the first (and as far as I know only) person to give **two** hour-talks at the **same** Oberwolfach combinatorics meeting. I remember these talks like they were given yesterday. They were definitely in the top ten talks that I have ever heard. What is so captivating about Dave's lecture-style is that unlike the rest of us, that try to state things in the most general setting (thereby completely obscuring the ideas), Dave went the other way, and made things as concrete as possible and actually had **numbers** in his talk, not general formulas. The formulas only came at the end, after the ideas and concepts were internalized.

On the way back, I was fortunate to share a train-cabin with Dave, and I asked him lots of questions, and thus started my love-hate relationship with the ASM conjecture.

Meanwhile, MRR found yet another conjecture, that the so-called *Totally Symmetric Self Complementary Plane Partitions* (TSSCPP) are also enumerated by the Robbins numbers. Around 1987, I had some rudimentary ideas for proving ASM=TSSCPP, but since the details seemed daunting, and proving them equal, while a nice result, would not yield the original ASM conjecture, I abandoned the attempt, focusing on more promising (i.e. easier) problems.

Then came George Andrews's amazing announcement, and proof, of the TSSCPP conjecture (that these too are enumerated by the Robbins numbers $1, 2, 7, 42, 429, \ldots$), and to my personal delight, George used, among other tricks of the trade, the so-called **Wilf-Zeilberger method**. Anyway, after George's amazing proof, it made sense to resume my attempts at proving ASM=TSSCPP, since it would entail, thanks to George Andrews, the original conjecture that ASMs are enumerated by the Robbins numbers.

After a few months, in December 1992, I produced the first version (about twenty pages), and sent it to yet another journal, *Journal of the Amer. Math. Soc.*, whose editor was Andrew Odlyzko, and sure enough, the 'anonymous' referee was Dave (as I immediately suspected and Dave confessed later). It was then that I started to 'hate' Dave. He very quickly found some *gaps*. I then fixed (or rather believed that I did) these gaps, and Dave found new ones. This process underwent three iterations, until Andrew Odlyzko sent me a *polite-but-firm* E-mail, that he kindly permitted me to quote.

```
From amo@research.att.com Wed May 11 06:35:00 1994
Date:  Wed, 11 May 94 06:34 EDT
To:  zeilberg@euclid.math.temple.edu
Subject:  paper for JAMS

Dear Doron,

The next message will contain a report on the latest version of your alternating sign
matrix conjecture.  As you can tell from the tone of the report, this referee is get-
ting a bit tired of dealing with your manuscript.  Please make sure that the next re-
vision is really solid before submitting it.

Best regards,
Andrew
```

But Dave was not the only one who was getting tired. Here is my reply to Andrew Odlyzko.

```
Dear Andrew,
Thanks for the report.  I will look at it and spend at most a day trying to ``fix''
the ``serious'' ``errors''.  If it would take me more than that, I will resubmit it
elsewhere.  I am sure that the proof is very robust.  No human proof can be ever com-
pletely formally correct, and because of the depth and complexity of this proof it
may be NP-complete to make it water-tight.  However, all the ``holes'', if holes there
are, should be a routine thing to fix, albeit time-consuming and extremely boring.

I will let you know in a day whether I plan to submit a 4th revised version.
```

```
I am getting tired too from the pedantry of the referee, who very possibly is David
Robbins, who has a vested interest in his conjecture enjoying a longer longevity than
it did, and is playing the filibuster.

Best Wishes, Doron
```

(In retrospect, my 'filibuster' accusation was wrong. Dave gave the same careful, 'I want to see all the details', treatment to *every* document that he refereed or reviewed, as I just found out yesterday in the short talks about him by his colleagues.)

And, finally, here is Andrew Odlyzko's defense of referee Dave.

```
From amo@research.att.com Wed May 11 12:58:27 1994
To:  zeilberg@math.temple.edu
Subject:  ASM
Dear Doron,
As far as the alternating sign matrix conjecture is concerned, I don't think it's just
a matter of pedantry, and the second referee did not think so either.  It won't do
to say that the proof is "robust." If that were acceptable, then the conjecture would
have been regarded as proved as soon as it was made.  After all, there was numeri-
cal evidence for it, and if God did not want such a beautiful result to be true, why
did He not provide a simple counterexample?  However, mathematics insists on a higher
level of proof than that.  Any respectable mathematical journal will surely want ei-
ther a proof that is verifiable by competent and careful referees, or a clear state-
ment of what the unproved assumptions are.

Best regards
Andrew
```

Then I got a brilliant idea to make the paper 'really solid'. It had to be 'pre-refereed'. So I first fixed the many gaps (some of them, in fairness to Dave and Andrew, were not so minor), and then I rewrote the paper (that turned out to be about seventy-two pages long) in a *structured*, *modular*, *distributed* fashion, organized into *lemmas*, *sublemmas*, *subsublemmas*, etc. (all the way to $(sub)^6$-lemmas), and asked for volunteers, each of whom would get one node of the proof-tree, and would only have to check that the assigned $(sub)^i$-lemma follows from all its children, which are $(sub)^{i+1}$-lemmas, or in the case of a leaf, just check its proof. In addition, I wrote a Maple package, `ROBBINS`, that accompanied the article, that checked *empirically* each statement made in the paper.

I sent the solicitation E-mail, that can be looked at:

`http://www.math.rutgers.edu/~zeilberg/asm/CHECKING` ,

to the 120 people in my E-mailing list (this was in the good old, pre-spam days), and got a very good response rate, more than two thirds. Their reports (on the draft) can be looked at:

`http://www.math.rutgers.edu/~zeilberg/asm/REPORTS` .

Once I fixed all the numerous (but minor) errors, and implemented the suggested improvements in exposition, I decided to forget JAMS, and to submit it to the special issue in honor of Dominique Foata's 60th birthday of the *Electronic Journal of Combinatorics*.

So Dave Robbins and Andrew Odlyzko's 'pedantic' insistence was for the best, since my ASM proof turned out to be a *paradigm* of formal correctness, thanks to its tree-structure and the many checkers. In my humble opinion, this innovative *format*, and the pioneering idea of *communal checking*, are even more important than the content of my article. I should also add that Dave Bressoud served as an independent checker for almost everything.

For the full story, in particular for Greg Kuperberg's shorter proof, and my proof of the Refined ASM conjecture, I refer you to Dave Bressoud's masterpiece *Proofs and Confirmations* (Cambridge Univ. Press and Math. Assoc. of Amer.) .

PostPostScript

Other 'outside work' (a term used by the IDA spooks to refer to unclassified math) gems of Dave include

• Continued Fractions over $GF(q)(x)$
• Expected assignments
• Extension of the classical formulas of Herron and Brahamgupta to areas of pentagons and beyond.
• Another beautiful explicit conjecture, about volumes of certain polytopes (joint with Clara Chan). This conjecture was also proved by yours truly, but this one only took one page (but it used the deep Morris-Selberg identity).

PostPostPostScript

In addition to the *short-term work*, *long-term work*, and *outside work*, Dave also had time to worry about the well-beings of our children, when he served, to almost everyone's satisfaction, on the Princeton Regional Board of Education for six years, three of them as President. When he was up for re-election, he asked me to be a signatory in a campaign ad that appeared in the local Princeton newspapers. I gladly agreed, and thanks to this, I got my name to be part of a very impressive list (perhaps the most illustrious list I ever belonged to). At the risk of sounding like a name-dropper, let me mention some of the signatories:

Bob Austin, John and Neta Bahcal, Luis Caffarelli, Freeman and Imre Dyson, Charles and Julie Fefferman, Chiara Nappi, Deborah Robbins, Miri and Nathan Seiberg, Lee Silver, Arthur Wightman, Ed Witten, and last but not least, *Doron Zeilberger.*

But my main point is that if all these great people supported Dave, it must say something, not only about Dave the mathematician, but about Dave the person.

Acknowledgement: Many thanks to Andrew Odlyzko for permission to quote from our E-correspondence, and to Lynne Butler and Clara Chan for organizing the great tribute for Dave while he was still able to enjoy it.