The Gift Exchange Problem

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Abstract

The aim of this paper is to solve the "gift exchange" problem: you are one of n players, and there are n wrapped gifts on display; when your turn comes, you can either choose any of the remaining wrapped gifts, or you can "steal" a gift from someone who has already unwrapped it, subject to the restriction that no gift can be stolen more than a total of σ times. The problem is to determine the number of ways that the game can be played out, for given values of σ and n. Several recurrences and explicit formulas are given for these numbers, although some open questions remain.

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1. The problem

The following game is sometimes played at parties. A number σ (typically 1 or 2) is fixed in advance. Each of the n guests brings a wrapped gift, the gifts are placed on a table (this is the "pool" of gifts), and slips of paper containing the numbers 1 to n are distributed randomly among the guests. The host calls out the numbers 1 through n in order.

When the number you have been given is called, you can either choose one of the wrapped (and so unknown) gifts remaining in the pool, or you can take (or "steal") a gift that some earlier person has unwrapped, subject to the restriction that no gift can be "stolen" more than a total of σ times.

If you choose a gift from the pool, you unwrap it and show it to everyone. If a person's gift is stolen from them, they immediately get another turn, and can either take a gift from the pool, or can steal someone else's gift, subject always to the limit of σ thefts per gift. The game ends when someone takes the last (nth) gift.

The problem is to determine the number of possible ways the game can be played out, for given values of σ and n.

For example, if $\sigma = 1$ and $n = 3$, with guests A, B, C and gifts numbered 1, 2, 3, there are 42 different scenarios, as follows. We write XN to indicate that guest X took gift N – it is always clear from the context whether the gift was stolen or taken from the pool. Also, provided we multiply the final answer by 6, we can assume that the gifts are taken from the pool in the order $1, 2, 3$.

There are then seven possibilities:

$$\begin{aligned}
&A1, B2, C3 \\
&A1, B2, C1, A3 \\
&A1, B2, C1, A2, B3 \\
&A1, B2, C2, B3 \\
&A1, B2, C2, B1, A3 \\
&A1, B1, A2, C3 \\
&A1, B1, A2, C2, A3
\end{aligned} \tag{1}$$

and so the final answer is $6 \cdot 7 = 42$.

If we continue to ignore the factor of $n!$ due to the order in which the gifts are selected from the pool, the number of scenarios for the case $\sigma = 1$ and $n = 1, 2, 3, 4, 5$ are $1, 2, 7, 37, 266$, respectively.

We noticed that these five terms matched the beginning of entry A001515 in [14], although indexed differently. The nth term of A001515 is defined as $y_n(1)$, where $y_n(x)$ is a Bessel polynomial ([8], [9], [13]), and for $n = 0, 1, 2, 3, 4$ the values are $1, 2, 7, 37, 266$, respectively. Although there was no mention of gift-swapping in that entry, one of the comments there provided enough of a hint to lead us to a complete solution of the general problem.

Comments on the rules

(i) If $\sigma = 1$ then once a gift has been stolen it can never be stolen again.
(ii) If $\sigma = 2$, and someone steals your gift, then if you wish you may immediately steal it back (provided you got it honestly!), and then it cannot be stolen again. Retrieving a gift in this way, although permitted by a strict interpretation of the rules, may be prohibited at real parties.
(iii) A variation of the game allows the last player to take *any* gift that has been unwrapped, regardless of how many times it has already been stolen, as an alternative to taking the last gift from the pool. This case only requires minor modifications of the analysis, and we will not consider it here.
(iv) We also ignore the complications caused by the fact that you brought (and wrapped) one of the gifts yourself, and so are presumably unlikely to choose it when your number is called.

2. Connection with partitions of labeled sets

Let $H_\sigma(n)$ be the number of scenarios with n gifts and a limit of σ steals, for $\sigma \geq 0, n \geq 1$. Then $H_\sigma(n)$ is a multiple of $n!$, and we write $H_\sigma(n) = n! G_\sigma(n-1)$, where in $G_\sigma(n-1)$ we assume that the gifts are taken from the pool in the order $1, 2, \ldots, n$. We write $n - 1$ rather than n as the argument of G_σ because the nth gift plays a special (and less important) role. This also simplifies the statement of Theorem 1.

In other words, $G_\sigma(n)$ is the number of scenarios when there are $n + 1$ gifts, with a limit of σ steals per gift, and the gifts are taken from the pool in the order $1, 2, \ldots, n + 1$.

As mentioned above, the sequence of values of $G_1(n)$ appeared to coincide with entry A001515 in [14]. One of the interpretations of that sequence (contributed by Robert A. Proctor on April 18, 2005) involved partitions of a labeled set into blocks, and this was enough of a hint to lead us to our first theorem.

We recall that the Stirling number of the second kind, $S_2(i,j)$, is the number of partitions of the labeled set $\{1,\ldots,i\}$ into j blocks ([6], [7]), while for $h \ge 1$ the h-restricted Stirling number of the second kind, $S_2^{(h)}(i,j)$, is the number of partitions of $\{1,\ldots,i\}$ into j blocks of size at most h ([3]-[5]).

Theorem 1. *For $\sigma \ge 0$ and $n \ge 0$,*

$$G_\sigma(n) = \sum_{k=n}^{(\sigma+1)n} S_2^{(\sigma+1)}(k,n)\,. \tag{2}$$

Proof. Equation (2) is an assertion about $G_\sigma(n)$, so we are now discussing scenarios where there are $n+1$ gifts. For $\sigma = 0$, $H_0(n+1) = (n+1)!$, so $G_0(n) = 1$, in agreement with $S_2^{(1)}(n,n) = 1$.

We may assume therefore that $\sigma \ge 1$. Let an "action" refer to a player choosing a gift γ, either by taking it from the pool or by stealing it from another player. Since we are now assuming that the gifts are taken from the pool in order, γ determines both the player and whether the action was to take a gift from the pool or to steal it from another player. So the scenario is fully specified simply by the sequence of γ values, recording which gift is chosen at each action. For example, the scenarios in (1) are represented by the sequences 123, 1213, 12123, 1223, 12213, 1123, 11223. Since the game ends as soon as the $(n+1)$st gift is selected, the number of actions is at least $n+1$ and at most $(\sigma+1)n+1$.

The sequence of γ values is therefore a sequence of integers from $\{1,\ldots,n+1\}$ which begins with 1, ends with $n+1$, where each number $i \in \{1,\ldots,n\}$ appears at least once and at most $\sigma+1$ times and $n+1$ appears just once, and in which the first i can appear only after $i-1$ has appeared. Conversely, any sequence with these properties determines a unique scenario.

Let k denote the length of the sequence with the last entry (the unique $n+1$) deleted. We map this shortened sequence to a partition of $[1,\ldots,k]$ into n blocks: the first block records the positions of the 1's, the second block records the positions of the 2's, …, and the nth block records the positions of the n's. Continuing the example, for the seven sequences above, the values of k and the corresponding partitions are as shown in Table 1.

Table 1: Values of k and partitions corresponding to the scenarios in (1).

k	partition
2	$1,2$
3	$13,2$
4	$13,24$
3	$1,23$
4	$14,23$
3	$12,3$
4	$12,34$

The number of such partitions is precisely $S_2^{(\sigma+1)}(k,n)$. Since the mapping from sequences to partitions is completely reversible, the desired result follows. ■

Remark. The sums $B(i) := \sum_j S_2(i,j)$ are the classical Bell numbers. The sums $\sum_j S_2^{(h)}(i,j)$ also have a long history [10], [11]. However, the sums $\sum_i S_2^{(h)}(i,j)$ mentioned in (2) do not seem to have studied before. Note that the limits in (2) are the natural limits on the summand k, and could be omitted.

To simplify the notation, and to put the most important variable first, let

$$E_\sigma(n,k) := S_2^{(\sigma+1)}(k,n)\,, \tag{3}$$

for $\sigma \ge 0$, $n \ge 0$, $k \ge 0$. In words, $E_\sigma(n,k)$ is the number of partitions of $\{1,\ldots,k\}$ into exactly n blocks of sizes in the range $[1,\ldots,\sigma+1]$.

For $n \ge 0$, $E_\sigma(n,k)$ is nonzero only for $n \le k \le (\sigma+1)n$. To avoid having to worry about negative arguments, we define $E_\sigma(n,k)$ to be zero if either n or k is negative. Then

$$G_\sigma(n) = \sum_{k=n}^{(\sigma+1)n} E_\sigma(n,k)\,. \tag{4}$$

Stirling numbers of the second kind satisfy many different recurrences and generating functions ([6, Chap. V]), and to a lesser extent this is also true for $E_\sigma(n,k)$. We begin with three general properties.

Theorem 2. *(i) Suppose $\sigma \ge 1$. Then $E_\sigma(n,k) = 0$ for $k < n$ or $k > (\sigma+1)n$, and otherwise, for $n \le k \le (\sigma+1)n$,*

$$E_\sigma(n,k) = \sum_{i=0}^{\sigma} \binom{k-1}{i} E_\sigma(n-1,k-1-i)\,. \tag{5}$$

(ii) For $\sigma \ge 0$, $n \ge 0$, $k \ge 0$,

$$E_\sigma(n,k) = \sum_{(a_1,\ldots,a_{\sigma+1})} \frac{k!}{a_1!a_2!\ldots a_{\sigma+1}!\,1!^{a_1}2!^{a_2}\cdots(\sigma+1)!^{a_{\sigma+1}}}\,, \tag{6}$$

where the sum is over all $(\sigma+1)$-tuples of nonnegative integers $(a_1,\ldots,a_{\sigma+1})$ satisfying

$$\begin{aligned} a_1 + a_2 + a_3 \cdots + a_{\sigma+1} &= n\,, \\ a_1 + 2a_2 + 3a_3 \cdots + (\sigma+1)a_{\sigma+1} &= k\,. \end{aligned} \tag{7}$$

(iii) The numbers $E_\sigma(n,k)$ have the exponential generating function

$$\sum_{n=0}^{\infty} \sum_{k=n}^{(\sigma+1)n} E_\sigma(n,k) x^n \frac{y^k}{k!} = \exp\left[x\left(y + \frac{y^2}{2!} + \cdots + \frac{y^{\sigma+1}}{(\sigma+1)!}\right)\right]. \tag{8}$$

Proof. (i) This is an analog of the "vertical" recurrence for the Stirling numbers ([6, Eq. [3c], p. 209]). The idea of the proof is to take a partition of $[1,\ldots,k]$, remove the block containing k, and renumber the remaining parts. (ii) Here a_i is the number of blocks of size i in the partition. This follows by standard counting arguments (cf. [6, Th. B, p. 205]). (iii) This is an analog of the "vertical" generating function for the Stirling numbers ([6, Eq. [2b], p. 206]), and follows directly from (i). ∎

The recurrence in Theorem 2(i) makes it easy to compute as many values of $E_\sigma(n,k)$ as one wishes. Tables 3 through 7 show the initial values of $E_1(n,k)$ through $E_5(n,k)$, and Table 8 gives the initial values of $G_\sigma(n)$ for $\sigma = 0$ through 8.

3. The case $\sigma = 1$

In the case when a gift can be stolen at most once, from Theorem 2 we have the recurrence

$$E_1(n,k) = E_1(n-1,k-1) + (k-1)E_1(n-1,k-2)\,, \tag{9}$$

for $n \le k \le 2n$, with $E_1(n,k) = 0$ for $k < n$ and $k > 2n$; the explicit formula

$$E_1(n,k) = \frac{k!}{(2n-k)!\;(k-n)!\;2^{k-n}}\,, \tag{10}$$

for $n \le k \le 2n$; and the generating function

$$\sum_{n=0}^{\infty}\sum_{k=n}^{2n} E_1(n,k)\; x^n \frac{y^k}{k!} = e^{x(y+y^2/2)}\,. \tag{11}$$

It follows from (4) that

$$\begin{aligned} G_1(n) &= \sum_{k=n}^{2n} \frac{k!}{(2n-k)!\;(k-n)!\;2^{k-n}} \\ &= \sum_{i=0}^{n} \frac{(n+i)!}{(n-i)!\;i!\;2^i}\,. \end{aligned} \tag{12}$$

Equation (12) shows that the sequence $G_1(n)$ is indeed given by entry A001515 in [14]. That entry gives (mostly without proof) several other properties of these numbers, taken from various sources, notably Grosswald [8]. We collect some of these properties in the next theorem. Property (iii) is especially interesting, since the following sections will be concerned with attempts to generalize it to larger values of σ. We recall from [8] that the Bessel polynomial $y_n(z)$ is given by

$$y_n(z) := \sum_{i=0}^{n} \frac{(n+i)! z^i}{(n-i)!\;i!\;2^i}\,. \tag{13}$$

Also ${}_2F_0$ and (later) ${}_2F_1$ denote hypergeometric functions.

Theorem 3. *(i)*

$$G_1(n) = y_n(1)\,. \tag{14}$$

(ii)

$$G_1(n) = {}_2F_0\left[\begin{matrix} n+1, -n \\ - \end{matrix}\;;\; -\tfrac{1}{2}\right]. \tag{15}$$

(iii)

$$G_1(n) = (2n-1)G_1(n-1) + G_1(n-2)\,. \tag{16}$$

for $n \ge 2$*, with* $G_1(0) = 1, G_1(1) = 2$.

(iv)

$$\sum_{n=0}^{\infty} G_1(n)\frac{x^n}{n!} = \frac{e^{1-\sqrt{1-2x}}}{\sqrt{1-2x}}\,. \tag{17}$$

(v)

$$G_1(n) \sim \frac{e(2n)!}{n!2^n} \text{ as } n \to \infty\,. \tag{18}$$

Proof. (i) and (ii) are immediate consequences of (12).
(iii) We give three proofs of (16). (First proof.) Equation (16) follows from one of the recurrences for Bessel polynomials ([8, Eq. (7), p. 18], [9]). (Second proof.) Alternatively, it is easy to verify from (10) that

$$E_1(n,k) = (2n-1)E_1(n-1,k-2) + E_1(n-2,k-2)\,. \tag{19}$$

Our conventions about negative arguments make it unnecessary to put any restrictions on the range over which (19) holds. By summing (19) on k we obtain (16). (Third proof.) The third proof is combinatorial. We will show the equivalent statement that for $n \geq 3$,

$$G_1(n) = G_1(n-2) + G_1(n-1) + 2(n-1)G_1(n-1)\,. \tag{20}$$

We can build a partition counted in $G_1(n)$ in three ways. (A) Take a partition P into $n-2$ parts and adjoin two parts of size 1, $\{x\}$ and $\{y\}$, say, where x, y are elements not in P. This gives $G_1(n-2)$ partitions. (B) Take a partition P into $n-1$ parts and adjoin a part $\{x,y\}$ of size 2. This gives $G_1(n-1)$ partitions. (C) Let P be a partition into $n-1$ parts and let S be one of the parts. If $S=\{u\}$ is a singleton, then

$$P \setminus S \cup \{u,x\} \cup \{y\} \text{ and } P \setminus S \cup \{u,y\} \cup \{x\}$$

are two partitions into n parts. If $S=\{u,v\}$ is a pair, then

$$P \setminus S \cup \{u,x\} \cup \{v,y\} \text{ and } P \setminus S \cup \{u,y\} \cup \{v,x\}$$

are two partitions into n parts. So in either case the pair P, S gives rise to two partitions into n parts. There are $n-1$ choices for S, so in all we obtain $2(n-1)G_1(n-1)$ partitions. The argument is clearly reversible, and so (20) and hence (16) follow.
(iv) Let

$$\begin{aligned}\mathcal{G}_1(x) &:= \sum_{n=0}^{\infty} G_1(n)\frac{x^n}{n!} \\ &= 1 + 2x + 7\frac{x^2}{2!} + 37\frac{x^3}{3!} + 266\frac{x^4}{4!} + \cdots\,.\end{aligned}$$

By multiplying (16) by $x^n/n!$ and summing on n from 2 to ∞ we obtain the differential equation

$$\mathcal{G}_1''(x) = 3\mathcal{G}_1'(x) + 2x\mathcal{G}_1''(x) + \mathcal{G}_1(x)\,. \tag{21}$$

Then the right-hand side of (17) is the unique solution of (21) which satisfies $\mathcal{G}_1(0)=1$, $\mathcal{G}_1'(0)=2$.
(v) This follows from (12), since the terms $i=n-1$ and $i=n$ dominate the sum (see also [8, Eq. (1), p. 124]). ■

4. The case $\sigma = 2$

In the case when a gift can be stolen at most once, the problem, as we saw in the previous section, turned out to be related to the values of Bessel polynomials, and the principal sequence, $G_1(n)$, had been studied before. For $\sigma \geq 2$, we appear to be in new territory—for one thing, the sequences $G_2(n), G_3(n), \ldots$ were not among the 140,000 existing sequences in [14].

These sequences can be computed using Theorem 2. From (4), (6) we have:

$$G_\sigma(n) = \sum_{k=n}^{(\sigma+1)n} \sum_{(a_1,\ldots,a_{\sigma+1})} \frac{k!}{a_1!a_2!\ldots a_{\sigma+1}!\,1!^{a_1}2!^{a_2}\cdots(\sigma+1)!^{a_{\sigma+1}}}\,, \tag{22}$$

where the inner sum is over all $(\sigma+1)$-tuples of nonnegative integers $(a_1,\ldots,a_{\sigma+1})$ satisfying (7). This may be rewritten as a sum of multinomial coefficients:

$$G_\sigma(n) \;=\; \frac{1}{n!}\sum_{i_1=1}^{\sigma+1}\sum_{i_2=1}^{\sigma+1}\cdots\sum_{i_n=1}^{\sigma+1}\binom{i_1+i_2+\cdots+i_n}{i_1,\; i_2,\; \cdots\;,\; i_n}\,, \tag{23}$$

where i_r is the size of the rth part.

We naturally tried to find analogs of the various parts of Theorem 3 that would hold for $\sigma \geq 2$. Let us begin with the simplest result, the asymptotic behavior. This is directly analogous to Theorem 3(v).

Theorem 4. *For fixed* $\sigma \geq 1$,

$$G_\sigma(n) \;\sim\; \frac{e((\sigma+1)n)!}{n!(\sigma+1)!^n} \text{ as } n \to \infty\,. \tag{24}$$

Sketch of proof. The two terms corresponding to $\{k=(\sigma+1)n, a_{\sigma+1}=n, \text{ other } a_i=0\}$ and $\{k=(\sigma+1)n-1, a_{\sigma+1}=n-1, a_\sigma=1, \text{ other } a_i=0\}$ dominate the right-hand side of (22), and are both equal to $((\sigma+1)n)!/(n!(\sigma+1)!^n)$. Dividing the sum by this quantity gives a converging sum, in which a subset of terms approach $1+1+1/2!+1/3!+...$, while the others vanish as $n\to\infty$. ■

Concerning Theorem 3(i), we do not know if there is a generalization of Bessel polynomials whose value gives (22) for $\sigma \geq 2$.

As for Theorem 3(ii), there is a relationship with hypergeometric functions in the case $\sigma = 2$. From (6) we have

$$\begin{aligned} E_2(n,k) \;&=\; \sum_{c=\max\{0,k-2n\}}^{\lfloor (k-n)/2\rfloor} \frac{k!}{(2n-k+c)!(k-n-2c)!c!\,2^{k-n-c}3^c} \\ &=\; \sum_{c=\max\{0,\eta-n\}}^{\lfloor \eta/2\rfloor} \frac{k!}{(n-\eta+c)!(\eta-2c)!c!\,2^{\eta-c}3^c}\,, \end{aligned} \tag{25}$$

where $\eta = k-n$ (this is the "excess" of k over n).

Theorem 5. *(i) Let* $\eta = k-n$.
If $\eta \leq n$ *then*

$$E_2(n,k) = \frac{(n+\eta)!}{\eta!(n-\eta)!2^\eta}\; {}_2F_1\left[\begin{matrix} -\eta/2, -\eta/2+1/2 \\ n-\eta+1 \end{matrix}\; ;\; \tfrac{8}{3}\right]. \tag{26}$$

If $\eta \geq n$ *then*

$$E_2(n,k) = \frac{(\eta+n)!}{(2n-\eta)!(\eta-n)!2^n 3^{\eta-n}}\; {}_2F_1\left[\begin{matrix} -n+\eta/2, -n+\eta/2+1/2 \\ \eta-n+1 \end{matrix}\; ;\; \tfrac{8}{3}\right]. \tag{27}$$

(ii)

$$\begin{aligned} G_2(n) \;&=\; \sum_{\eta=0}^{n-1} \frac{(n+\eta)!}{\eta!(n-\eta)!2^\eta}\; {}_2F_1\left[\begin{matrix} -\eta/2, -\eta/2+1/2 \\ n-\eta+1 \end{matrix}\; ;\; \tfrac{8}{3}\right] \\ &+\; \sum_{\eta=n}^{2n} \frac{(n+\eta)!}{(2n-\eta)!(\eta-n)!2^n 3^{\eta-n}}\; {}_2F_1\left[\begin{matrix} -n+\eta/2, -n+\eta/2+1/2 \\ \eta-n+1 \end{matrix}\; ;\; \tfrac{8}{3}\right]. \end{aligned} \tag{28}$$

Proof. (i) follows from (25) using the standard rules for converting sums of products of factorials to hypergeometric functions (cf. [1]), and (ii) follows from (4). ■

We can now state the main theorem of this section, which gives analogs of (19) and (16).

Theorem 6. *(i)*

$$\begin{aligned} E_2(n,k) &= (9n^2-9n+2)E_2(n-1,k-3)/2 - 5E_2(n-1,k-1)/2 \\ &+ (9n^2-36n+35)E_2(n-2,k-4)/2 + 6(n-1)E_2(n-2,k-3) - 3E_2(n-2,k-2)/2 \\ &+ 3(2n-5)E_2(n-3,k-4) + 5E_2(n-3,k-3)/2 + 5E_2(n-4,k-4)/2\,. \end{aligned} \tag{29}$$

(ii)

$$\begin{aligned} G_2(n) &= (9n^2-9n-3)G_2(n-1)/2 \\ &+ (9n^2-24n+20)G_2(n-2)/2 \\ &+ (6n-25/2)G_2(n-3) + 5G_2(n-4)/2\,, \end{aligned} \tag{30}$$

for $n \ge 4$*, with* $G_2(0)=1$*,* $G_2(1)=3$*,* $G_2(2)=31$*,* $G_2(3)=18252$*.*

Proof. (ii) Eq. (30) follows by summing (29) on k, just as (16) followed from (19).

(i) We give two proofs of (29). The first proof uses (26), (27) and Gauss's contiguity relations for hypergeometric functions ([2, §2.1.2], [15, §14.7]). There are nine $E_2(i,j)$ terms in (29), and each of them is given by either (26) or (27), depending on the relationship between i and j. This means that six separate cases must be considered, according to whether $k \ge 2n+1$, $k = 2n, 2n-1, 2n-2, 2n-3$ or $k \le 2n-4$. We give the details just for the first case, the other cases being very similar. Assuming then that $k \ge 2n+1$, (26) applies to all nine $E_2(i,j)$ terms in (29). Writing $\eta = k-n$ as before, and replacing the final argument $\frac{8}{3}$ in the hypergeometric functions by a new variable z, we must show that the expression

$$\begin{aligned} &\frac{(\eta+n)!}{(\eta-n)!(2n-\eta)!2^n3^{\eta-n}}\ {}_2F_1\left[\begin{matrix}\eta/2-n,\eta/2-n+1/2\\ \eta-n+1\end{matrix};\ z\right] \\ -\ &\frac{9n^2-9n+2}{2}\ \frac{(\eta+n-3)!}{(\eta-n-1)!(2n-\eta)!2^{n-1}3^{\eta-n-1}}\ {}_2F_1\left[\begin{matrix}\eta/2-n,\eta/2-n+1/2\\ \eta-n\end{matrix};\ z\right] \\ +\ &\frac{5}{2}\ \frac{(\eta+n-1)!}{(\eta-n+1)!(2n-\eta-2)!2^{n-1}3^{\eta-n+1}}\ {}_2F_1\left[\begin{matrix}\eta/2-n+1,\eta/2-n+3/2\\ \eta-n+2\end{matrix};\ z\right] \\ -\ &\frac{9n^2-36n+35}{2}\ \frac{(\eta+n-4)!}{(\eta-n)!(2n-\eta-2)!2^{n-2}3^{\eta-n}}\ {}_2F_1\left[\begin{matrix}\eta/2-n+1,\eta/2-n+3/2\\ \eta-n+1\end{matrix};\ z\right] \\ -\ &6(n-1)\ \frac{(\eta+n-3)!}{(\eta-n+1)!(2n-\eta-3)!2^{n-2}3^{\eta-n+1}}\ {}_2F_1\left[\begin{matrix}\eta/2-n+3/2,\eta/2-n+2\\ \eta-n+2\end{matrix};\ z\right] \\ +\ &\frac{3}{2}\ \frac{(\eta+n-2)!}{(\eta-n+2)!(2n-\eta-4)!2^{n-2}3^{\eta-n+2}}\ {}_2F_1\left[\begin{matrix}\eta/2-n+2,\eta/2-n+5/2\\ \eta-n+3\end{matrix};\ z\right] \\ -\ &3(2n-5)\frac{(\eta+n-4)!}{(\eta-n+2)!(2n-\eta-5)!2^{n-3}3^{\eta-n+2}}\ {}_2F_1\left[\begin{matrix}\eta/2-n+5/2,\eta/2-n+3\\ \eta-n+3\end{matrix};\ z\right] \\ -\ &\frac{5}{2}\ \frac{(\eta+n-3)!}{(\eta-n+3)!(2n-\eta-6)!2^{n-3}3^{\eta-n+3}}\ {}_2F_1\left[\begin{matrix}\eta/2-n+3,\eta/2-n+7/2\\ \eta-n+4\end{matrix};\ z\right] \\ -\ &\frac{5}{2}\ \frac{(\eta+n-4)!}{(\eta-n+4)!(2n-\eta-8)!2^{n-4}3^{\eta-n+4}}\ {}_2F_1\left[\begin{matrix}\eta/2-n+4,\eta/2-n+9/2\\ \eta-n+5\end{matrix};\ z\right] \end{aligned} \tag{31}$$

vanishes when $z = \frac{8}{3}$: Using Gauss's contiguity relations, the nine hypergeometric functions in (31) can all be expressed as linear combinations of just two of them. The computer algebra program Maple 11 simplifies[1] the above expression to

$$\frac{(\eta+n-4)!(3z-8)}{324(\eta-n+1)!(2n-\eta-2)!2^n 3^{\eta-n} z^3 (z-1)^3} \left(\phi_1 \, {}_2F_1 \left[\begin{matrix} \eta/2-n+1, \eta/2-n+3/2 \\ \eta-n+2 \end{matrix} ; \; z \right] \right.$$
$$\left. + \;\; \phi_2 \, {}_2F_1 \left[\begin{matrix} \eta/2-n, \eta/2-n+1/2 \\ \eta-n+1 \end{matrix} ; \; z \right] \right), \qquad (32)$$

where ϕ_1 and ϕ_2 are polynomials in z of degrees 6 and 5 respectively, with coefficients which are polynomials in n and η. Since the exact values of ϕ_1 and ϕ_2 are not important for the argument, we relegate them to Tables 9 and 10 in the Appendix. The above expression clearly vanishes for $z = \frac{8}{3}$, which proves the desired result.

Second proof. Let

$$D_2(n,k,c) := \frac{k!}{(2n-k+c)!(k-n-2c)!c!\,2^{k-n-c}3^c} \qquad (33)$$

denote the first summand in (25). We look for a recurrence of the form

$$\sum_{r=0}^{4}\sum_{s=0}^{4}\sum_{t=0}^{4} C(r,s,t) D_2(n+r,k+s,c+t) \;=\; 0\,, \qquad (34)$$

where the coefficients $C(r,s,t)$ depend on n but not on k or c, with the property that when summed on c it collapses to the appropriately shifted version of (29), which is:

$$\begin{aligned} & E_2(n+4,k+4) - (9n^2+63n+110)E_2(n+3,k+1)/2 + 5E_2(n+3,k+3)/2 \\ & - \;(9n^2+36n+35)E_2(n+2,k)/2 + 6(n+3)E_2(n+2,k+1) + 3E_2(n+2,k+2)/2 \\ & - \;3(2n+3)E_2(n+1,k) - 5E_2(n+1,k+1)/2 - 5E_2(n,k)/2 \;=\; 0\,. \end{aligned} \qquad (35)$$

For this we used the method of Sister Mary Celine Fasenmyer, exactly as described in §4.1 of [12]. A Maple 11 program found that there is a solution to (34) in which the coefficients $C(n,k,c)$ involve five free parameters, and there is a two-parameter solution which collapses to (35) when summed on c. The simplest solution (obtained from Maple's solution by setting both free parameters to zero) is the following. All the $C(r,s,t)$ are zero except for the following 19 terms:

$C(0,0,1) = -8$,	$C(2,1,1) = -9$,
$C(0,0,2) = 7$,	$C(2,1,2) = 3$,
$C(0,0,3) = -3/2$,	$C(2,2,1) = 6$,
$C(1,0,1) = -18$,	$C(2,2,2) = -6$,
$C(1,0,2) = 15$,	$C(2,2,3) = 3/2$,
$C(1,0,3) = -3$,	$C(3,1,0) = -9$,
$C(1,1,1) = -4$,	$C(3,3,1) = 5$,
$C(1,1,2) = 3/2$,	$C(3,3,2) = -5/2$,
$C(2,0,0) = -9$,	$C(4,4,1) = 1$,
$C(2,0,1) = 9$.	

[1] We don't actually know how Maple obtains (32), but the result is consistent with the use of Gauss's relations.

It is easy to verify that this collapses to (35) when summed on c. ■

Is there a combinatorial proof for (30)? We do not know.

We discovered (30) by experiment, using Theorem 6 to suggest the leading term. (Note that if $r(n)$ denotes the right-hand side of (24), then $r(n)/r(n-1) = (9n^2-9n+2)/2$.) We also discovered a second recurrence, which is independent of (30):

$$\begin{aligned}(n-2)G_2(n) = {} & n(9n^2-27n+17)G_2(n-1)/2 \\ & + (6n^2-15n+13/2)G_2(n-2) \\ & + (5n-5)G_2(n-3)/2\,,\end{aligned} \tag{36}$$

for $n \ge 3$, with $G_2(0) = 1$, $G_2(1) = 3$, $G_2(2) = 31$. In view of (28), this is equivalent to a complicated identity involving hypergeometric functions. We did not find a proof, but Doron Zeilberger has kindly informed us that he was able to derive (36) by applying the method of "creative telescoping" ([12, Chap. 6], [17], [18]) to (33) and using a modified version of his Maple program "MultiZeilberger".

5. The case $\sigma \ge 3$

For $\sigma \ge 3$ we have not found any connections between $G_\sigma(n)$ and generalized Bessel polynomials or hypergeometric functions, and we do not have proofs for the recurrences that we have discovered.

However, we do know that recurrences for $G_\sigma(n)$ and $E_\sigma(n,k)$ always exist. This follows from Wilf and Zeilberger's Fundamental Theorem for Multivariate Sums ([12, Theorem 4.5.1], [16]).

Theorem 7. *(i) For $\sigma \ge 1$, there is a number $\delta \ge 0$ such that $E_\sigma(n,k)$ satisfies a recurrence of the form*

$$\sum_{i=0}^{\delta}\sum_{j=0}^{\delta} C_{i,j}^{(E)}(n) E_\sigma(n-i,k-j) = 0 \text{ for all } n\,, \tag{37}$$

where the coefficients $C_{i,j}^{(E)}(n)$ are polynomials in n with coefficients depending on i and j.
(ii) For $\sigma \ge 1$, there is a number $\delta \ge 0$ such that $G_\sigma(n)$ satisfies a recurrence of the form

$$\sum_{i=0}^{\delta}\sum_{j=0}^{\delta} C_{i}^{(G)}(n) G_\sigma(n-i) = 0 \text{ for all } n\,, \tag{38}$$

where the coefficients $C_{i}^{(G)}(n)$ are polynomials in n with coefficients depending on i.

Proof. (ii) As usual, Eq. (38) follows by summing (37) on k. (i) We will use the case $\sigma = 3$ to illustrate the proof, the general case being similar. We know from (6) that

$$E_3(n,k) = \sum_{a,b,c,d} \frac{k!}{a!b!c!d!\,2^b 6^c 24^d}\,, \tag{39}$$

where the sum is over all 4-tuples of nonnegative integers (a,b,c,d) satisfying

$$\begin{aligned} a+b+c+d &= n\,, \\ a+2b+3c+4d &= k\,. \end{aligned}$$

In other words,

$$\frac{E_3(n,k)}{2^n} = \sum_{c,d} \frac{k!}{(2n-k+c+2d)!(k-n-2c-3d)!c!d!\,2^{k-c}3^{c+d}}\,, \tag{40}$$

where now the sum is over all values of c and d for which the summand is defined. This summand is a "holonomic proper-hypergeometric term", in the sense of [16], and it follows from the Fundamental Theorem in that paper that $E_3(n,k)/2^n$ and hence $E_3(n,k)$ satisfies a recurrence of the desired form. Similarly, in the general case, we write the summand in $E_\sigma(n,k)$ as a function of $n, k, a_3, \ldots, a_{\sigma+1}$, again obtaining a holonomic proper-hypergeometric term. ■

We conjecture, but do not have a proof, that a stronger result holds, namely that recurrences always exist in which the leading terms $C_{0,0}^{(E)}$ and $C_0^{(G)}$ are both 1, as in (16), (19), (29), (30), (41), (42) and Table 11. (The recurrence guaranteed by Theorem 7 may well look more like (36), with a nontrivial coefficient on the leading term.)

For $\sigma = 3, 4$ and 5, we have found recurrences for $E_\sigma(n,k)$ and $G_\sigma(n)$ with leading coefficient 1, although we do not have proofs that they are correct. The following are our conjectured recurrences for $G_3(n)$ and $G_4(n)$:

$$\begin{aligned} G_3(n) &= (32n^3/3 - 16n^2 + 10n/3 - 49/6)G_3(n-1) \\ &+ (48n^3 - 236n^2 + 1157n/3 - 650/3)G_3(n-2) \\ &+ (80n^3 - 382n^2 + 641n - 511)G_3(n-3)/3 \\ &+ (64n^3/3 - 218n^2 + 2696n/3 - 7915/6)G_3(n-4) \\ &+ (56n^2 - 490n + 6853/6)G_3(n-5) \\ &+ (56n - 1703/6)G_3(n-6) \\ &+ 58G_3(n-7)/3\,, \end{aligned} \tag{41}$$

$$\begin{aligned} G_4(n) &= (625\,n^4 - 1250\,n^3 + 625\,n^2 - 300\,n - 543)G_4(n-1)/24 \\ &+ (27500\,n^4 - 184000\,n^3 + 447500\,n^2 - 473075\,n + 180003)G_4(n-2)/72 \\ &+ (336875\,n^4 - 2546500\,n^3 + 7679675\,n^2 - 12016800\,n + 8048577)G_4(n-3)/864 \\ &+ (4833125\,n^4 - 77581625\,n^3 + 476892700\,n^2 - 1304291160\,n + 1325759504)G_4(n-4)/2592 \\ &+ (1700625\,n^4 + 28316750\,n^3 - 605973450\,n^2 + 3123850885\,n - 5033477363)G_4(n-5)/7776 \\ &+ (2670000\,n^4 - 64380500\,n^3 + 704577200\,n^2 - 3610058445\,n + 6818722190)G_4(n-6)/7776 \\ &+ (2002500\,n^4 - 51976000\,n^3 + 517392050\,n^2 - 2252744530\,n + 3561765885)G_4(n-7)/7776 \\ &+ (9078000\,n^3 - 209915400\,n^2 + 1640828980\,n - 4301927039)G_4(n-8)/7776 \\ &+ (5393400\,n^2 - 91413680\,n + 390747263)G_4(n-9)/2592 \\ &+ (1593990\,n - 14522219)G_4(n-11)/972 \\ &+ 310343G_4(n-11)/648\,. \end{aligned} \tag{42}$$

The recurrence for $G_5(n)$ is similar but more complicated, and we do not state it here. The recurrence for $E_3(n,k)$ is given in the Appendix (see Table 11). We also omit the recurrences for for $E_4(n,k)$ and $E_5(n,k)$, which are even more complicated.

Inspection of these recurrences for $\sigma \le 5$ has led us to some conjectures about their general structure. First, if δ denotes the "depth" of the recurrence, as in (37), (38), then the initial values of

δ for both $G_\sigma(n)$ and $E_\sigma(n,k)$ appear to be as shown in Table 2, that is, it appears that these both recurrences have depth $\delta = \binom{n+1}{2}+1$ (sequence A000124 of [14]). Second, we make the following

Table 2: Depth δ of recurrences for $G_\sigma(n)$ and $E_\sigma(n,k)$.

σ	0	1	2	3	4	5
δ	1	2	4	7	11	16

conjectures[2] about the coefficients in the putative recurrence for $E_\sigma(n,k)$. We write this recurrence as

$$\sum_{i=0}^{\delta}\sum_{j=0}^{\delta} C_{i,j}^{(E)}(n)E_\sigma(n-i,k-j) = 0\,, \tag{43}$$

where $\delta = \binom{n+1}{2}+1, C_{0,0}^{(E)}(n) = 1$. Then we believe that $C_{i,j}(n) = 0$ if $j > \binom{n+1}{2}+1$, or $j < i$, or $(i < \sigma$ and $j > \binom{n+1}{2}+1-((\sigma+1-i)^2-\sigma-i-1)/2)$. Furthermore, the degree of $C_{i,j}(n)$ as a polynomial in n is $\le \min\{\sigma, j-i\}$.

6. Open questions

We collect here some of the questions that we have mentioned. (i) The case $\sigma = 1$ corresponds to values of Bessel polynomials; is there a notion of generalized Bessl polynomial that could be applied for larger values of σ? (ii) The case $\sigma = 2$ can be described using hypergeometric functions; is there a notion of generalized hypergeometric function that could be applied for larger values of σ? (iii) Is there a combinatorial proof of (30)? (iv) Is the conjecture following Theorem 7 concerning the existence of recurrences with leading coefficient 1 true? (v) Find proofs that the recurrences (41) and (42) are correct. (vi) Establish the conjectures about the general form of the recurrences for $G_\sigma(n)$ and $E_\sigma(n,k)$ that are mentioned at the end of §5 (this includes question (iv) as a special case).

7. Acknowledgment

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Appendix

This Appendix collects various tables and multi-line formulas that would otherwise have disrupted the flow of the text.

Notation: $E_\sigma(n,k)$ is the number of partitions of $\{1,\ldots,k\}$ into exactly n blocks of sizes in the range $[1,\ldots,\sigma+1]$. Also $G_\sigma(n)$ is the number of scenarios when there are $n+1$ gifts, with a limit of σ steals per gift, and the gifts are taken from the pool in the order $1,2,\ldots,n+1$.

Table 3: Values of $E_1(n,k)$. The array itself appears in several versions in [14]: see for example A001498, A144299, A144331; the row sums give A001515, the column sums give A000085 (cf. [10], [11]).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	...
1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	...
2	0	0	1	3	3	0	0	0	0	0	0	0	0	0	...
3	0	0	0	1	6	15	15	0	0	0	0	0	0	0	...
4	0	0	0	0	1	10	45	105	105	0	0	0	0	0	...
5	0	0	0	0	0	1	15	105	420	945	945	0	0	0	...
6	0	0	0	0	0	0	1	21	210	1260	4725	10395	10395	0	...
7	0	0	0	0	0	0	0	1	28	378	3150	17325	62370	135135	...
8	0	0	0	0	0	0	0	0	1	36	630	6930	51975	945945	...
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...

Table 4: Values of $E_2(n,k)$. The array itself appears in several versions in [14]: see for example A144385, A144399, A144402; the row sums give A144416, the column sums give A001680 (cf. [10]).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	...
1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	...
2	0	0	1	3	7	10	10	0	0	0	0	0	0	0	...
3	0	0	0	1	6	25	75	175	280	280	0	0	0	0	...
4	0	0	0	0	1	10	65	315	1225	3780	9100	15400	15400	0	...
5	0	0	0	0	0	1	15	140	980	5565	26145	102025	323400	800800	...
6	0	0	0	0	0	0	1	21	266	2520	19425	125895	695695	3273270	...
7	0	0	0	0	0	0	0	1	28	462	5670	56595	478170	3488485	...
8	0	0	0	0	0	0	0	0	1	36	750	11550	144375	1531530	...
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...

Table 5: Values of $E_3(n,k)$. The array itself appears in several versions in [14]: see A144643, A144644, A144645; the row sums give A144508, the column sums give A001681 (cf. [10]).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	…
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	…
1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	…
2	0	0	1	3	7	15	25	35	35	0	0	0	0	0	…
3	0	0	0	1	6	25	90	280	770	1855	3675	5775	5775	0	…
4	0	0	0	0	1	10	65	350	1645	6930	26425	90475	275275	725725	…
5	0	0	0	0	0	1	15	140	1050	6825	39795	211750	1033725	4629625	…
6	0	0	0	0	0	0	1	21	266	2646	22575	172095	1198120	7702695	…
7	0	0	0	0	0	0	0	1	28	462	5880	63525	609840	5335330	…
8	0	0	0	0	0	0	0	0	1	36	750	11880	158235	1861860	…
…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…

Table 6: Values of $E_4(n,k)$. The array itself appears in several versions in [14]: see A151338, A151509, A151511; the row sums give A144509, the column sums give A110038 (cf. [10]).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	…
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	…
1	0	1	1	1	1	1	0	0	0	0	0	0	0	0	…
2	0	0	1	3	7	15	31	56	91	126	126	0	0	0	…
3	0	0	0	1	6	25	90	301	938	2737	7455	18711	41811	81081	…
4	0	0	0	0	1	10	65	350	1701	7686	32725	132055	505351	1824823	…
5	0	0	0	0	0	1	15	140	1050	6951	42315	241780	1310925	6782776	…
6	0	0	0	0	0	0	1	21	266	2646	22827	179025	1309000	9054045	…
7	0	0	0	0	0	0	0	1	28	462	5880	63987	626472	5677672	…
8	0	0	0	0	0	0	0	0	1	36	750	11880	159027	1897896	…
…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…

Table 7: Values of $E_5(n, k)$. The array itself appears in several versions in [14]: see A151359, A151511, A151512; the row sums give A149187, the column sums give A148092 (cf. [10]).

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	...
1	0	1	1	1	1	1	1	0	0	0	0	0	0	0	...
2	0	0	1	3	7	15	31	63	119	210	336	462	462	0	...
3	0	0	0	1	6	25	90	301	966	2989	8925	25641	70455	183183	...
4	0	0	0	0	1	10	65	350	1701	7770	33985	143605	588511	2341339	...
5	0	0	0	0	0	1	15	140	1050	6951	42525	246400	1370985	7383376	...
6	0	0	0	0	0	0	1	21	266	2646	22827	179487	1322860	9294285	...
7	0	0	0	0	0	0	0	1	28	462	5880	63987	627396	5713708	...
8	0	0	0	0	0	0	0	0	1	36	750	11880	159027	1899612	...
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...

Table 8: Number $G_\sigma(n)$ of scenarios when there are $n + 1$ gifts, a limit of σ steals per gift and the gifts are taken from the pool in the order $1, 2, \ldots, n$. (The array itself is entry A144512 in [14]; the rows give A001515, A144416, A144508, A144509, A149187; the columns A048775, A144511, A144662, A147984.)

n	0	1	2	3	4	5	...
$G_0(n)$	1	1	1	1	1	1	...
$G_1(n)$	1	2	7	37	266	2431	...
$G_2(n)$	1	3	31	842	45296	4061871	...
$G_3(n)$	1	4	121	18252	7958726	7528988476	...
$G_4(n)$	1	5	456	405408	1495388159	15467641899285	...
$G_5(n)$	1	6	1709	9268549	295887993624	34155922905682979	...
$G_6(n)$	1	7	6427	216864652	60790021361170	79397199549271412737	...
$G_7(n)$	1	8	24301	5165454442	12845435390707724	191739533381111401455478	...
$G_8(n)$	1	9	92368	124762262630	2774049143394729653	476872353039366288373555323	...
...	...	...	...	...	...	...	...

Table 9: The polynomial ϕ_1 mentioned in the first proof of Theorem 6.

$$
\begin{aligned}
\phi_1 = {} & (486 + 729\,\eta n - 2349\,n + 2916\,n^2 - 162\,\eta - 729\,n^3 - 729\,n^2\eta)z^6 \\
& + (-45\,n + 306\,\eta - 1080\,n^2 + 207\,n^3 + 180\,\eta^3 - 216\,\eta^2 - 324\,\eta^2 n \\
& \quad - 1539\,\eta n + 1647\,n^2\eta - 54)z^5 \\
& + (3441\,n^3 - 4023\,n^2\eta + 24621\,n - 348\,\eta^3 - 16884\,n^2 + 1260\,\eta^2 n \\
& \quad - 10650\,\eta - 1800\,\eta^2 + 13203\,\eta n - 9054)z^4 \\
& + (341\,n^3 + 948\,\eta^2 n + 8614\,\eta - 3359\,n + 261\,n^2\eta + 984\,\eta^2 \\
& \quad - 6081\,\eta n - 484\,\eta^3 + 3270)z^3 \\
& + (-35572\,n - 36712\,\eta n - 4092\,n^3 + 13512\,\eta^2 + 2572\,\eta^3 + 14952 \\
& \quad + 25892\,\eta + 11164\,n^2\eta - 9244\,\eta^2 n + 22472\,n^2)z^2 \\
& + (11200\,\eta^2 n - 20160\,\eta^2 - 3200\,\eta^3 - 21120 - 24320\,n^2 + 45760\,\eta n \\
& \quad + 4160\,n^3 - 38080\,\eta + 42560\,n - 12160\,n^2\eta)z \\
& + (7680 + 7680\,n^2 + 14080\,\eta - 15360\,\eta n - 3840\,\eta^2 n + 1280\,\eta^3 \\
& \quad - 14080\,n - 1280\,n^3 + 7680\,\eta^2 + 3840\,n^2\eta)
\end{aligned}
$$

Table 10: The polynomial ϕ_2 mentioned in the first proof of Theorem 6.

$$
\begin{aligned}
\phi_2 = {} & (27\,\eta^2 + 216\,\eta n - 189\,\eta + 324 + 189\,n^2 - 675\,n)z^5 \\
& + (-9\,\eta^2 - 9\,n^2 - 504\,\eta n + 495\,\eta + 9\,n + 486)z^4 \\
& + (-15\,\eta^2 + 600\,\eta n - 1191\,\eta - 789\,n^2 - 3672 + 3399\,n)z^3 \\
& + (-243\,\eta^2 + 408\,\eta n - 555\,\eta - 303\,n^2 - 978 + 1155\,n)z^2 \\
& + (560\,\eta^2 - 1360\,\eta n + 3040\,\eta - 3440\,n + 720\,n^2 + 3840)z \\
& + (-320\,\eta^2 + 640\,\eta n - 1600\,\eta - 320\,n^2 - 1920 + 1600\,n)\,.
\end{aligned}
$$

Table 11: The conjectured recurrence for $E_3(n,k)$. Summing both sides on k gives (41).

$$
\begin{aligned}
E_3(n,k) = {} & (32n^3/3 - 16n^2 + 22n/3 - 1)E_3(n-1,k-4) \\
& - (4n + 3/2)E_3(n-1,k-2) \\
& - 17E_3(n-1,k-1)/3 \\
\\
& + (16n^3 - 88n^2 + 159n - 189/2)E_3(n-2,k-6) \\
& + (32n^3 - 176n^2 + 914n/3 - 497/3)E_3(n-2,k-5) \\
& + (28n^2 - 66n + 46)E_3(n-2,k-4) \\
& + (-12n + 29/2)E_3(n-2,k-3) \\
& - 17E_3(n-2,k-2) \\
\\
& + (-16n^3/3 + 152n^2/3 - 479n/3 + 1001/6)E_3(n-3,k-7) \\
& + (32n^3 - 262n^2 + 2218n/3 - 4255/6)E_3(n-3,k-6) \\
& + (84n^2 - 382n + 1247/3)E_3(n-3,k-5) \\
& + (16n - 47/3)E_3(n-3,k-4) \\
& - 28E_3(n-3,k-3) \\
\\
& + (64n^3/3 - 302n^2 + 4154n/3 - 12427/6)E_3(n-4,k-7) \\
& + (84n^2 - 562n + 2858/3)E_3(n-4,k-6) \\
& + (76n - 187)E_3(n-4,k-5) \\
& - 41E_3(n-4,k-4)/3 \\
\\
& + (56n^2 - 574n + 4352/3)E_3(n-5,k-7) \\
& + (84n - 651/2)E_3(n-5,k-6) \\
& + 17E_3(n-5,k-5) \\
\\
& + (56n - 1877/6)E_3(n-6,k-7) \\
& + 29E_3(n-6,k-6) \\
\\
& + 58E_3(n-7,k-7)/3\,.
\end{aligned}
$$