An Inelegant (but Short(!)) Proof of a Major Index Theorem of Garsia and Gessel (Verbose Version)

Doron ZEILBERGER[1]

Recall Adriano Garsia's favorite notation: $\chi(A) = 1$ if A is true, and $\chi(A) = 0$ if A is false. For example $\chi(MathIsFun) = 1$, $\chi(HeleneBarceloIsAGoodEditor) = 0$. Also recall that the *major index* of any list of integers $\pi = (\pi_1, \ldots, \pi_n)$ (in particular a permutation) is defined by $maj(\pi) := \sum_{i=1}^{n-1} i\chi(\pi_i > \pi_{i+1})$. Let $\theta = \theta_1 \ldots \theta_a$ and $\pi = \pi_1 \ldots \pi_b$ be two disjoint lists of distinct integers. Let $S(\theta, \pi)$ be the set of (a+b)!/(a!b!) mergings (suffles) of θ and π. Let $(q)_n := (1-q)(1-q^2)\cdots(1-q^n)$. Moti Novick[N] has recently found an elegant bijective proof of the following

Theorem (Garsia and Gessel[GG]):

$$\sum_{\sigma \in S(\theta,\pi)} q^{maj(\sigma)} = \frac{(q)_{a+b}}{(q)_a (q)_b} q^{maj(\theta)+maj(\pi)} \quad . \qquad (Moti)$$

In this short note I will present an *inelegant*, induction proof, by proving, more generally:

Lemma: Let $S_1(\theta, \pi)$ be the subset of $S(\theta, \pi)$ whose last entry is θ_a, and let $S_2(\theta, \pi)$ be the subset of $S(\theta, \pi)$ whose last entry is π_b, then

$$\sum_{\sigma \in S_1(\theta,\pi)} q^{maj(\sigma)} = \frac{(q)_{a+b-1}}{(q)_{a-1} (q)_b} q^{maj(\theta)+maj(\pi)+b\chi(\pi_b>\theta_a)} \quad , \qquad (Adriano)$$

$$\sum_{\sigma \in S_2(\theta,\pi)} q^{maj(\sigma)} = \frac{(q)_{a+b-1}}{(q)_a (q)_{b-1}} q^{maj(\theta)+maj(\pi)+a\chi(\pi_b<\theta_a)} \quad . \qquad (Ira)$$

Note that adding-up $(Adriano)$ and (Ira) gives $(Moti)$. Let's call the left-sides of $(Adriano)$ and (Ira) $F_1(a, b)$ and $F_2(a, b)$ respectively, and let's call their right-sides $G_1(a, b)$ and $G_2(a, b)$ respectively. It is immediate that $(F_1, F_2) = (G_1, G_2)$ when $a = 0$ or $b = 0$, and the fact that $(F_1, F_2) = (G_1, G_2)$ for all $a, b \geq 0$ follows from the fact that both $(X_1, X_2) = (F_1, F_2)$ and $(X_1, X_2) = (G_1, G_2)$ satisfy the recurrence

$$X_1(a, b) = q^{(a+b-1)\chi(\theta_{a-1}>\theta_a)} X_1(a-1, b) + q^{(a+b-1)\chi(\pi_b>\theta_a)} X_2(a-1, b)$$

$$X_2(a, b) = q^{(a+b-1)\chi(\theta_a>\pi_b)} X_1(a, b-1) + q^{(a+b-1)\chi(\pi_{b-1}>\pi_b)} X_2(a, b-1) \quad .$$

The proof for (F_1, F_2) is left to the reader. Let's prove it for (G_1, G_2). We will only prove the first identity. The second one is also left to the reader.

[1] Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA. `zeilberg at math dot rutgers dot edu` , `http://www.math.rutgers.edu/~zeilberg/` . First version: March 25, 2009. This verbose version: March 31, 2009. Supported in part by the NSF.

We have to prove that

$$G_1(a,b) - q^{(a+b-1)\chi(\theta_{a-1}>\theta_a)} G_1(a-1,b) - q^{(a+b-1)\chi(\pi_b>\theta_a)} G_2(a-1,b) = 0 \quad . \qquad (HaImEfes)$$

By definition:

$$G_1(a,b) = \frac{(q)_{a+b-1}}{(q)_{a-1}(q)_b} q^{maj(\theta_1\ldots\theta_a)+maj(\pi_1\ldots\pi_b)+b\chi(\pi_b>\theta_a)} \quad ,$$

$$G_2(a,b) = \frac{(q)_{a+b-1}}{(q)_a(q)_{b-1}} q^{maj(\theta_1\ldots\theta_a)+maj(\pi_1\ldots\pi_b)+a\chi(\pi_b<\theta_a)} \quad .$$

Replacing a by $a-1$ we have

$$G_1(a-1,b) = \frac{(q)_{a+b-2}}{(q)_{a-2}(q)_b} q^{maj(\theta_1\ldots\theta_{a-1})+maj(\pi_1\ldots\pi_b)+b\chi(\pi_b>\theta_{a-1})} \quad ,$$

$$G_2(a-1,b) = \frac{(q)_{a+b-2}}{(q)_{a-1}(q)_{b-1}} q^{maj(\theta_1\ldots\theta_{a-1})+maj(\pi_1\ldots\pi_b)+(a-1)\chi(\pi_b<\theta_{a-1})} \quad .$$

Substituting in the left side of $(HaImEfes)$, we have to prove:

$$\frac{(q)_{a+b-1}}{(q)_{a-1}(q)_b} q^{maj(\theta_1\ldots\theta_a)+maj(\pi_1\ldots\pi_b)+b\chi(\pi_b>\theta_a)}$$

$$-q^{(a+b-1)\chi(\theta_{a-1}>\theta_a)} \cdot \frac{(q)_{a+b-2}}{(q)_{a-2}(q)_b} q^{maj(\theta_1\ldots\theta_{a-1})+maj(\pi_1\ldots\pi_b)+b\chi(\pi_b>\theta_{a-1})}$$

$$-q^{(a+b-1)\chi(\pi_b>\theta_a)} \cdot \frac{(q)_{a+b-2}}{(q)_{a-1}(q)_{b-1}} q^{maj(\theta_1\ldots\theta_{a-1})+maj(\pi_1\ldots\pi_b)+(a-1)\chi(\pi_b<\theta_{a-1})} = 0$$

Dividing by $\frac{(q)_{a+b-2}}{(q)_{a-1}(q)_b}$, we have to prove

$$(1-q^{a+b-1}) q^{maj(\theta_1\ldots\theta_a)+maj(\pi_1\ldots\pi_b)+b\chi(\pi_b>\theta_a)}$$

$$-q^{(a+b-1)\chi(\theta_{a-1}>\theta_a)} \cdot (1-q^{a-1}) q^{maj(\theta_1\ldots\theta_{a-1})+maj(\pi_1\ldots\pi_b)+b\chi(\pi_b>\theta_{a-1})}$$

$$-q^{(a+b-1)\chi(\pi_b>\theta_a)} \cdot (1-q^b) q^{maj(\theta_1\ldots\theta_{a-1})+maj(\pi_1\ldots\pi_b)+(a-1)\chi(\pi_b<\theta_{a-1})} = 0 \quad .$$

But

$$maj(\theta_1\ldots\theta_a) = maj(\theta_1\ldots\theta_{a-1}) + (a-1)\chi(\theta_{a-1}>\theta_a) \quad ,$$

so we have to prove:

$$(1-q^{a+b-1}) q^{maj(\theta_1\ldots\theta_{a-1})+(a-1)\chi(\theta_{a-1}>\theta_a)+maj(\pi_1\ldots\pi_b)+b\chi(\pi_b>\theta_a)} -$$

$$q^{(a+b-1)\chi(\theta_{a-1}>\theta_a)} \cdot (1-q^{a-1}) q^{maj(\theta_1\ldots\theta_{a-1})+maj(\pi_1\ldots\pi_b)+b\chi(\pi_b>\theta_{a-1})}$$

$$-q^{(a+b-1)\chi(\pi_b>\theta_a)} \cdot (1-q^b) q^{maj(\theta_1\ldots\theta_{a-1})+maj(\pi_1\ldots\pi_b)+a\chi(\pi_b<\theta_{a-1})} = 0$$

Dividing both sides by $q^{maj(\theta_1\ldots\theta_{a-1})+maj(\pi_1\ldots\pi_b)}$, we have to prove

$$(1-q^{a+b-1}) q^{(a-1)\chi(\theta_{a-1}>\theta_a)+b\chi(\pi_b>\theta_a)} - q^{(a+b-1)\chi(\theta_{a-1}>\theta_a)} \cdot (1-q^{a-1}) q^{b\chi(\pi_b>\theta_{a-1})}$$

$$-q^{(a+b-1)\chi(\pi_b>\theta_a)} \cdot (1-q^b)q^{(a-1)\chi(\pi_b<\theta_{a-1})} = 0$$

Combining powers of q, we have to prove:

$$(1-q^{a+b-1})q^{(a-1)\chi(\theta_{a-1}>\theta_a)+b\chi(\pi_b>\theta_a)} - (1-q^{a-1})q^{b\chi(\pi_b>\theta_{a-1})+(a+b-1)\chi(\theta_{a-1}>\theta_a)}$$

$$-(1-q^b)q^{(a-1)\chi(\pi_b<\theta_{a-1})+(a+b-1)\chi(\pi_b>\theta_a)} = 0$$

We have **six** cases to consider according to the relative rankings of $\theta_{a-1}, \theta_a, \pi_b$.

Case 123: $\theta_{a-1} < \theta_a < \pi_b$. We have to prove

$$(1-q^{a+b-1})q^b - (1-q^{a-1})q^b - (1-q^b)q^{(a+b-1)} = 0 \quad .$$

Dividing by q^b, we have to prove

$$(1-q^{a+b-1}) - (1-q^{a-1}) - (1-q^b)q^{(a-1)} = 0 \quad .$$

Expanding, we have, to prove

$$1 - q^{a+b-1} - 1 + q^{a-1} - q^{a-1} + q^{a+b-1} = 0 \quad ,$$

and this is indeed true.

Case 132: $\theta_{a-1} < \pi_b < \theta_a$. We have to prove

$$(1-q^{a+b-1})q^0 - (1-q^{a-1})q^b - (1-q^b)q^0 = 0$$

Expanding, we get:

$$1 - q^{a+b-1} - q^b + q^{a+b-1} - 1 + q^b = 0 \quad ,$$

and this is indeed true.

Case 213: $\theta_a < \theta_{a-1} < \pi_b$:

$$(1-q^{a+b-1})q^{(a-1)+b} - (1-q^{a-1})q^{b+(a+b-1)} - (1-q^b)q^{a+b-1} = 0 \quad ,$$

Dividing by q^{a+b-1}, we have to prove

$$(1-q^{a+b-1}) - (1-q^{a-1})q^b - (1-q^b) = 0 \quad ,$$

Expanding, we have to prove

$$1 - q^{a+b-1} - q^b + q^{a+b-1} - 1 + q^b = 0 \quad ,$$

and this is indeed true.

Case 231: $\theta_a < \pi_b < \theta_{a-1}$. We have to prove

$$(1-q^{a+b-1})q^{(a-1)+b} - (1-q^{a-1})q^{a+b-1} - (1-q^b)q^{a-1+(a+b-1)} = 0 \quad .$$

Dividing by q^{a+b-1}, we have to prove

$$(1-q^{a+b-1}) - (1-q^{a-1}) - (1-q^b)q^{a-1} = 0 \quad .$$

Expanding, we have to prove

$$1 - q^{a+b-1} - 1 + q^{a-1} - q^{a-1} + q^{a+b-1} = 0 \quad ,$$

and this is indeed true.

Case 312: $\pi_b < \theta_{a-1} < \theta_a$:

$$(1-q^{a+b-1})q^0 - (1-q^{a-1})q^0 - (1-q^b)q^{a-1} = 0 \quad ,$$

Expanding, we have to prove

$$1 - q^{a+b-1} - 1 + q^{a-1} - q^{a-1} + q^{a+b-1} = 0 \quad ,$$

and this is indeed true.

Case 321: $\pi_b < \theta_a < \theta_{a-1}$:

$$(1-q^{a+b-1})q^{a-1} - (1-q^{a-1})q^{a+b-1} - (1-q^b)q^{a-1} = 0 \quad .$$

Dividing by q^{a-1}, we have to prove

$$(1-q^{a+b-1}) - (1-q^{a-1})q^b - (1-q^b) = 0 \quad .$$

Expanding, we have to prove

$$1 - q^{a+b-1} - q^b + q^{a+b-1} - 1 + q^b = 0 \quad ,$$

and this is indeed true. □

Remarks: **1.** Moti Novick's elegant bijection also preserves equidistribution over Inverse Descent Classes. **2.** It would be interesting to bijectify the above inductive proof, along the lines of [MZ], and see if the resulting bijection is identitical, or similar, to Moti Novick's elegant bijection.

References:

[GG] Adriano M. Garsia and Ira Gessel, *Permutation statistics and partitions*, Advances in Mathematics **31**(1979), 288-305.

[MZ] Philip Matchett Wood and Doron Zeilberger, *A Translation Method for Finding Combinatorial Bijections*, to appear in Annals of Combinatorics.
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[N] Moti Novick, *A Bijective proof of a major index theorem of Garsia and Gessel*, preprint, Hebrew University of Jerusalem.