

# An Inelegant (but Short(!)) Proof of a Major Index Theorem of Garsia and Gessel

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Recall Adriano Garsia's favorite notation:  $\chi(A) = 1$  if  $A$  is true, and  $\chi(A) = 0$  if  $A$  is false. For example  $\chi(\text{MathIsFun}) = 1$ ,  $\chi(\text{HeleneBarceloIsAGoodEditor}) = 0$ . Also recall that the *major index* of any list of integers  $\pi = (\pi_1, \dots, \pi_n)$  (in particular a permutation) is defined by  $\text{maj}(\pi) := \sum_{i=1}^{n-1} i\chi(\pi_i > \pi_{i+1})$ . Let  $\theta = \theta_1 \dots \theta_a$  and  $\pi = \pi_1 \dots \pi_b$  be two disjoint lists of distinct integers. Let  $S(\theta, \pi)$  be the set of  $(a+b)!/(a!b!)$  mergings (suffles) of  $\theta$  and  $\pi$ . Let  $(q)_n := (1-q)(1-q^2) \dots (1-q^n)$ . Moti Novick[N] has recently found an elegant bijective proof of the following

**Theorem** (Garsia and Gessel[GG]):

$$\sum_{\sigma \in S(\theta, \pi)} q^{\text{maj}(\sigma)} = \frac{(q)_{a+b}}{(q)_a (q)_b} q^{\text{maj}(\theta) + \text{maj}(\pi)} . \quad (\text{Moti})$$

In this short note I will present an *inelegant*, induction proof, by proving, more generally:

**Lemma:** Let  $S_1(\theta, \pi)$  be the subset of  $S(\theta, \pi)$  whose last entry is  $\theta_a$ , and let  $S_2(\theta, \pi)$  be the subset of  $S(\theta, \pi)$  whose last entry is  $\pi_b$ , then

$$\sum_{\sigma \in S_1(\theta, \pi)} q^{\text{maj}(\sigma)} = \frac{(q)_{a+b-1}}{(q)_{a-1} (q)_b} q^{\text{maj}(\theta) + \text{maj}(\pi) + b\chi(\pi_b > \theta_a)} , \quad (\text{Adriano})$$

$$\sum_{\sigma \in S_2(\theta, \pi)} q^{\text{maj}(\sigma)} = \frac{(q)_{a+b-1}}{(q)_a (q)_{b-1}} q^{\text{maj}(\theta) + \text{maj}(\pi) + a\chi(\pi_b < \theta_a)} . \quad (\text{Ira})$$

Note that adding-up (*Adriano*) and (*Ira*) gives (*Moti*). Let's call the left-sides of (*Adriano*) and (*Ira*)  $F_1(a, b)$  and  $F_2(a, b)$  respectively, and let's call their right-sides  $G_1(a, b)$  and  $G_2(a, b)$  respectively. It is immediate that  $(F_1, F_2) = (G_1, G_2)$  when  $a = 0$  or  $b = 0$ , and the fact that  $(F_1, F_2) = (G_1, G_2)$  for all  $a, b \geq 0$  follows from the fact that both  $(X_1, X_2) = (F_1, F_2)$  and  $(X_1, X_2) = (G_1, G_2)$  satisfy the recurrence

$$\begin{aligned} X_1(a, b) &= q^{(a+b-1)\chi(\theta_{a-1} > \theta_a)} X_1(a-1, b) + q^{(a+b-1)\chi(\pi_b > \theta_a)} X_2(a-1, b) \\ X_2(a, b) &= q^{(a+b-1)\chi(\theta_a > \pi_b)} X_1(a, b-1) + q^{(a+b-1)\chi(\pi_{b-1} > \pi_b)} X_2(a, b-1) . \end{aligned}$$

**Remarks:** **1.** Moti Novick's elegant bijection also preserves equidistribution over Inverse Descent Classes. **2.** It would be interesting to bijectify the above inductive proof, along the lines of [MZ], and see if the resulting bijection is identical, or similar, to Moti Novick's elegant bijection.

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## References:

[GG] Adriano M. Garsia and Ira Gessel, *Permutation statistics and partitions*, Advances in Mathematics **31**(1979), 288-305.

[MZ] Philip Matchett Wood and Doron Zeilberger, *A Translation Method for Finding Combinatorial Bijections*, to appear in Annals of Combinatorics.

Available from <http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/trans-method.html> .

[N] Moti Novick, *A Bijective proof of a major index theorem of Garsia and Gessel*, <http://arxiv.org/abs/0906.0377> .