The Number of 1...d-Avoiding Permutations of Length d+r for SYMBOLIC d but Numeric r

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Dedicated to Ira Martin GESSEL (b. April 9, 1951), on his millionth$_2$ birthday

Preface: How many permutations are there of length googol+30 avoiding an increasing subsequence of length googol?

This number is way too big for our physical universe, but the number of permutations of length googol+30 that *contain* at least one increasing subsequence of length googol is

3769987628815905643852921525646105664146833823621994801456991357113502936781270538054719048039675278
0769193354371721350001524610578097700045972792823897290959624203896101981952929640805170129282073883
4740018807571147534091229951249435913149171795302592312477456091277812321956212802204785578598020255
5625008802850838455586257402947256848380647181479993222566420025908679106917004348077812428261510240
6340176300585397517990032393036653951304924586489968650809789292291489270968710994809677050176596751
0725956202350750841376095024046396844968511243494784162014881795337835528626142808150073111101283361
0980701571937952824136796425017224636196853995950587943259043687431653922927840572864396105085190223
2582799067810378389890635196322425667467335158890500820128768331750855469963050322432973194472331947
0989825934469696079344723053679001130033667827524966034661782064851068214182454731365743413486729730
0631055444127725930013792836515384850702346797298406803049230145697433567004811555984158378611125895
0145768901348725550726037527669812626353266837685037397408862767082018239579392663024131792105407280
4788720840618514463465035392103884394981202007834724110094447116613439128758285044269471808502083275
6629374247928521501786839409853287740758570056230853738462527374534709641735458487560816949365616486
0695626913029699922648102091615529949414940648588048836485372758775808743231365617459515329190972398
7074543946415578728439994306071279654045146432379557541358408978156863172980419720839292761025261752
6805876626590163265795592248178664681630980893821587688413815206609216082514983787883386977226071420
2164914772899359257896142217770029448259674099398651935724695993066814650508527071447556115011374722 1
2088787004775335817731620626335692795572945875468655064443263468768028202797640277277248383611710547 3
4814561150922815451047200040413061463978092641713732993973246572201468056490283993082430683492041454 5
1387475360005525209200113681457132938458732558246848788724439529524558541918864679276425283215996202 9
6941164954437213105323538743944687543469879373512141279640023696573258448468721998289835514598029197 7
8626923448613597311256425024700723913528035577571226795472601903389377137876277742366919657529517451 2
9645258766972572614483274037178282230800617053191009926567814148362251714401410771621701009838383996 8
8450780459024472066740659392956413459154780579363446852393445432890675675870120976547151488057237075 9
0908433173621630228907517716180640208908388998946733429336657675542373884509955262827926993717691558 8
5942783587044453980064448005282163092233177993702322865630527297415993197736506481784936186090944530 1
0812014057743669000714070157059948417686104746105282677474489924674666690926457806707624392345308856 1
9669877806921776719438294136573211203941287971353199159831767568250543984542462560043822507697311658 6
4913021330851479972883076463717212900406561190747561040171300879097289140902036265874194650989183216 5
7701667667006001209610998909380382010865003885220777565531701133543218588330720970852694358826481897 7

3775738149186073685934586558285596632901636818878886042883326839132327059391308990152857750191809745 6
3487912142476276560621310123468845009650614775925658273562207923751954793994347093016618292164580402 7
1254279814338864116761417830119059817479387880694430162532210993791275528220779177902246600447925840 8
2446294959276134988131654342203869918382647352510707580950827477809341316822096398440902856636229390 0
4021541588241935864951867435541480109504741382604082456634512978942603922184208879705298143957373669 6
5022330793088649449089550661242226637700975872048802255877951342510043234892643042766512594498769308 9
4224575112270628402898275433738688545939162654357055516205161266436378837328045722669166090867956953 9
2716308156251990403004593327493174233201870456895707500259180589455710602937342719975864491923386968 8
5903842897776980002129651552219483587717759774043798881299174958483572178675529350262014933898703122 2
3251822518408158990271446362436501824274759908263581759373772458033768880934255069534236693503642535 4
9188091443537667487643227020476441406556138221242510025369533668010935357887804140526277263813912479 2
8321640648394196028626519959966325451252664262353889631883841776653646129270593661149306259085397802 4
1862926623393421168173669371424135263438461510848532070094781148761874414915822566817516932438525928 4
5563436340937294481843784242150745917626033404675889463006327603959116662310009262655062833600709070 6
4341332664779779937712263184388203605477211162014805937750522978535620225925004722918738657674699944 9
4740534790765914361805057941708749765216546018547704334563663220497822600180042427352634146022024254 8
68372879917906503008302949451445090553172508996790329329093550087454853933917873519408569488210748631 8
79883374585250820777287677645800280443076699166062637606763797777023540421219334461005282376299007226 5
78307082023454514148089887463748610689381677459821466400715603888673197538425720238 2 .

Hence the number of permutations of length googol+30 *avoiding* an increasing subsequence of length googol is $(googol + 30)!$ *minus* the above small number.

Counting the "Bad Guys"

Recall that thanks to Robinson-Schensted ([Rob][Sc]), the number of permutations of length n that do **not** contain an increasing subsequence of length d is given by

$$G_d(n) := \sum_{\substack{\lambda \vdash n \\ \#rows(\lambda) < d}} f_\lambda^2 \quad ,$$

where λ denotes a typical *Young diagram*, and f_λ is the number of *Standard Young tableaux* whose *shape* is λ.

Hence the number of permutations of length n that **do** contain an increasing subsequence of length d is

$$B_d(n) := \sum_{\substack{\lambda \vdash n \\ \#rows(\lambda) \geq d}} f_\lambda^2 \quad .$$

Since the total number of permutations of length n is $n!$ ([B]), if we know how to find $B_d(n)$, we would know immediately $G_d(n) = n! - B_d(n)$, at least if we leave $n!$ alone as a factorial, rather than spell it out.

Recall that the *Hook Length formula* (see [Wiki]) tells you that if λ is a Young diagram then

$$f_\lambda = \frac{n!}{\prod_{c \in \lambda} h(c)} \quad ,$$

where the product is over all the n cells of the Young diagram, and the *hook-lenght*, $h(c)$, of a cell $c = (i, j)$, is $(\lambda_i - i) + (\lambda'_j - j) + 1$, where λ' is the *conjugate* diagram, where the rows become columns and vice-versa.

Let r be a fixed integer, then for *symbolic* d, valid for $d \geq r - 1$, any Young diagram with at least d rows, and with $d + r$ cells, can be written, for some Young diagram $\mu = (\mu_1, \ldots, \mu_r)$, with $\leq r$ cells, (where we add zeros to the end if the number of parts of μ is less than r) as

$$\lambda = (1 + \mu_1, \ldots, 1 + \mu_r, 1^{d-r+r'}) \quad ,$$

where $r' = r - |\mu|$. For such a shape λ, with *at least* d rows,

$$\prod_{c\in\lambda} h(c) = \left(\prod_{c\in\mu} h(c) \right) \cdot ((d + r' + \mu_1)(d + r' - 1 + \mu_2) \cdots (d + r' - r + 1 + \mu_r)) \cdot (d - r + r')! \quad .$$

Hence f_λ, that is $(d+r)!$ divided by the above, is a certain specific number times a certain polynomial in d. Since, for a specific, *numeric*, r, there are only *finitely* many Young diagrams with at most r cells, the computer can find all of them, compute the polynomial corresponding to each of them, square it, and add-up all these terms, getting an *explicit* **polynomial** expression, in the variable d, for $B_d(d + r)$, the number of permutations of length $d + r$ that *contain* an increasing subsequence of length d. As we said above, from this we can find $G_d(d + r) = (d + r)! - B_d(d + r)$, valid for *symbolic* $d \geq r - 1$.

$\mathbf{B_d(d + r)}$ for r from 0 to 30

$$B_d(d) = 1 \quad ,$$

$$B_d(d + 1) = d^2 + 1 \quad ,$$

$$B_d(d + 2) = \frac{1}{2}\, d^4 + d^3 + \frac{1}{2}\, d^2 + d + 3 \quad ,$$

$$B_d(d + 3) = \frac{1}{6}\, d^6 + d^5 + \frac{5}{3}\, d^4 + \frac{2}{3}\, d^3 + \frac{19}{6}\, d^2 + \frac{31}{3}\, d + 11 \quad ,$$

$$B_d(d + 4) = \frac{1}{24}\, d^8 + \frac{1}{2}\, d^7 + \frac{25}{12}\, d^6 + \frac{19}{6}\, d^5 + \frac{29}{24}\, d^4 + 9\, d^3 + \frac{247}{6}\, d^2 + \frac{395}{6}\, d + 47 \quad ,$$

$$B_d(d+5) = \frac{1}{120}\, d^{10} + \frac{1}{6}\, d^9 + \frac{31}{24}\, d^8 + \frac{14}{3}\, d^7 + \frac{823}{120}\, d^6 + \frac{67}{30}\, d^5 + \frac{653}{24}\, d^4 + \frac{959}{6}\, d^3 + \frac{10459}{30}\, d^2 + \frac{3981}{10}\, d + 239 \quad .$$

For $B_d(d + r)$ for r from 6 up to 30, see
`http://www.math.rutgers.edu/~zeilberg/tokhniot/oGessel64a` .

Sequences

The sequence $G_3(n)$ is the greatest *celeb* in the kingdom of combinatorial sequences [the subject of an entire book([St]) by Ira Gessel's illustrious *academic father*, Richard Stanley], the super-famous **A000108** in Neil Sloane's legendary database ([Sl]). $G_4(n)$, while not in the same league as the Catalan sequence, is still moderately famous, **A005802**. $G_5(n)$ is **A047889**, $G_6(n)$ is

A047890, $G_7(n)$ is **A052399** , $G_8(n)$ is **A072131**, $G_9(n)$ is **A072132**, $G_{10}(n)$ is **A072133**, $G_{11}(n)$ is **A072167**, but $G_d(n)$ for $d \geq 12$ are absent (for a good reason, one must stop somewhere!). Also the *flattened version* of the *double-sequence*, $\{G_d(n)\}$, for $1 \leq d \leq n \leq 45$ is **A047887**. Using the polynomials $B_d(d+r)$, we computed the first $2d+1$ terms of $G_d(n)$ for $d \leq 30$. See `http://www.math.rutgers.edu/~zeilberg/tokhniot/oGessel64b` .

But this method can only go up to $2d+1$ terms of the sequence $G_d(n)$, and of course, the first $d-1$ terms are trivial, namely $d!$ (and the d-th term is $d!-1$). Can we find the first 100 terms (or whatever) for the sequences $G_d(n)$ for d up to 20, and beyond, **efficiently?**

Encore: Efficient Computer-Algebra Implementation of Ira Gessel's AMAZING Determinant Formula

Recall Ira Gessel's [G] famous expression for the generating function of $G_d(n)/n!^2$, *canonized* in the *bible* ([W], p. 996, Eq. (5)). Here it is:

$$\sum_{n \geq 0} \frac{G_d(n)}{n!^2} x^{2n} = \det(I_{|i-j|}(2x))_{i,j=1,\dots,d} \quad ,$$

in which $I_\nu(t)$ is (the modified Bessel function)

$$I_\nu(t) = \sum_{j=0}^{\infty} \frac{(\frac{1}{2}\, t)^{2j+\nu}}{j!(j+\nu)!} \quad .$$

Can we use this to compute the first 100 terms of, say, $G_{20}(n)$?

While computing *numerical* determinants is very fast, computing *symbolic* ones is a different story. First, do not get scared by the "infinite" power series. If we are only interested in the first N terms of $G_d(n)$, then it is safe to truncate the series up to t^{2N}, and take the determinant of a $d \times d$ matrix with *polynomial entries*. If you use the *vanilla* determinant in a computer-algebra system such as Maple, it would be very inefficient, since the degree of the determinant is much larger than $2N$. But a little cleverness can make things more efficient. The Maple package `Gessel64`, available free of charge from

`http://www.math.rutgers.edu/~zeilberg/tokhniot/Gessel64` ,

accompanying this article, has a procedure `SeqIra(k,N)` that computes the first N terms of $G_k(n)$, using a division-free algorithm (see [Rot]) over an appropriate ring to compute the determinant in Gessel's famous formula.

```
SeqIra:=proc(k,N) local ira,t,i,j, R:

  R := table():

  R[`0`] := 0:

  R[`1`] := 1:
```

```
  R[`+`] := `+`:

  R[`-`] := `-`:

  R[`*`] := proc(p, q): return add(coeff(p*q, t, i)*t**i, i=0..2*N): end:

  R[`=`] := proc(p, q): return evalb(p = q): end:

  ira:=expand(LinearAlgebra[Generic][Determinant][R](Matrix([seq([seq(Iv(abs(i-j),t,2*N),
                                                                  j=1..k-1)],
                                                               i=1..k-1)]))):

  [seq(coeff(ira,t,2*i)*i!**2,i=1..N)]:

end:
```

In the above code, procedure `Iv(v,t,N)` computes the truncated modified Bessel function that shows up in Gessel's determinant, and it is short enough to reproduce here:

```
Iv:=proc(v,t,N) local j:  add(t**(2*j+v)/j!/(j+v)!,j=0..trunc((N-v)/2)+1):  end:
```
.

Using this procedure, the first-named author computed (in 4507 seconds) the first 100 terms of each of the sequences $G_d(n)$ for $3 \leq d \leq 20$, and could have gone much further.

See `http://www.math.rutgers.edu/~zeilberg/tokhniot/oGessel64c` .

HAPPY 64th BIRTHDAY, IRA!

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