Yesterday, 06 December 2014, held the Putnam Math Competition throughout the U.S.A. One of the problems in the morning session is about a determinant evaluation.

**Problem A2.** Find a closed form for the determinant

$$\det \left( \frac{1}{\min(i,j)} \right)_{i,j=1}^{n}.$$ 

Of course, a careful elementary row and column expansions would yield the value $(-1)^{n-1} \frac{n}{n!^2}$. Herewith, we generalize and prove the result in a unified and simpler way.

**Generalization.** Suppose $a, b \in \mathbb{N}$. We have

$$\det \left( \frac{1}{x_{\min(i+a,j+b)}} \right)_{i,j=1}^{n} = \begin{cases} \frac{1}{\mathcal{x}_{\min(a+1,b+1)}} & \text{if } n = 1 \text{ and } a \neq b \\ 0 & \text{if } n > 1 \text{ and } a \neq b \\ \prod_{i=1}^{n-1} \frac{x_{i+a} - x_{i+a+1}}{x_{i+a}} & \text{if } a = b. \end{cases}$$

**Proof.** Denoting the left-hand side by $M_n(a, b)$. The proof can readily be executed (inductively) by the Dodgson’s Condensation method in the form

$$M_n(a, b) = M_{n-1}(a, b)M_{n-1}(a+1, b+1) - M_n(a+1, b)M_{n-1}(a, b+1).$$

The reader is advised to consider the 3 different cases, separately. □

The following identity appeared in a paper by T. Mansour and Y. Sun ("Dyck paths and partial Bell polynomials") where it is stated for $n = pk + \ell$ where $0 \leq \ell \leq k - 1$. We generalize and provide a proof with the WZ methodology.

**Proposition.** For any $n, k, \ell \in \mathbb{N}$, we have

$$\sum_{m=0}^{\min(n,p)} \binom{n+m\ell-nk}{m} \binom{p}{m} = \frac{n(p(\ell+1) + n - pk + 1)}{(n+1)(p+1)} \binom{n+p}{n}.$$ 

**Proof.** Divide the summand on the left side by the right-hand side to denote by $F(n, m)$. Zeilberger’s algorithm provides the recurrence $F(n+1, k) - F(n, k) = G(n, k + 1) - G(n, k)$ where $G(n, k) = R(n, k)F(n, k)$ with the rational function $R(n, k) = \frac{n(n+1)p}{Q(n, k)}$ as a certificate given by

$$P(n, k) = k - \ell + n + pn - nk + np + n^2 + 2\ell k - 2km - 2\ell km + 2n\ell + nm\ell - nkm - \ell p + kp - pl^2 + pk^2 + plm - pkm + p\ell^2 m + pk^2 m + 2p\ell k + k^2 m + \ell^2 m - k^2 - \ell^2 - 2\ell km,$$

and $Q(n, k) = (-p + m - 1)(-n - m\ell + km)(-n - p - 2 + kp + k - \ell \ell)(n + p + 1)$. □