PROOF OF A q-ANALOG OF A CONSTANT TERM IDENTITY CONJECTURED BY FORRESTER

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As always in q-theory, $(X;Q)_n$ will stand for the product $(1-X)(1-QX)...(1-Q^{n-1}X)$, and when the "base" Q is q, we will abbreviate $(X;q)_n$ to $(X)_n$. For any Laurent polynomial f in $x_1,...,x_n$, $CT(f)$ denotes the coefficient of $x_1^0..x_n^0$. Throughout this paper $t := q^a$, $s = q^b$, $u = q^c$.

Theorem(q-extension of (4.2) in [F2]):

$$CT \prod_{i=1}^{n}(1+yx_i)\prod_{i=1}^{n}(x_i)_b(\frac{q}{x_i})_c \prod_{1<=i<j<=n}(\frac{x_i}{x_j})_a(\frac{qx_j}{x_i})_a = \qquad (*)$$

$$\prod_{j=0}^{n-1}\frac{(q)_{b+c+ja}(q)_{(j+1)a}}{(q)_{b+ja}(q)_{c+ja}(q)_a}\sum_{r=0}^{n}\frac{(t;t)_n(u;t)_r(qs;t)_{n-r}}{(t;t)_r(t;t)_{n-r}(qs;t)_n}(-qy)^r \ ,$$

Proof: As in [Z], first use the Stembridge-Stanton trick, to transform (*) to the equivalent "anti-symmetric" version, let's call it (*'), in which $(\frac{qx_j}{x_i})_a$ is replaced by $(\frac{qx_j}{x_i})_{a-1}$, and the right side of (*) gets multiplied by $(1-t)/(t;t)_a$. Next expand the very first product on the left of (*'), into a sum of 2^n terms, and note that they are all *bad guys* (see [Z], p. 314), except for the $n+1$ terms $x_1...x_r y^r$, $r=0,...,n$, the corresponding constant terms of which are evaluated by [Z]'s eq. (5.1). QED

Historical Notes: The special case $q=1$, $b=1$,$c=0$, of the above theorem was conjectured in [F1]. Shaun Cooper[C] formulated a conjecture for the general q-case, with still $b=1$, $c=0$. After Peter Forrester received a preliminary version of the present paper, that proved Cooper's conjectured q-extension of his original conjecture, he also received the preprint [K1], from which[F2] he was able to derive the special case $q=1$ of the above theorem. Forrester told us that the general theorem should follow from [K2] in an analogous way.

The present proof is shorter (even with [Z]) than Kaneko's proof, and entirely elementary, but Kaneko proves an even more general theorem.

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References:

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[K1] J. Kaneko, *Selberg integrals and hypergeometric functions associated with Jack polynomials*, preprint.

[K2] ", *q-Selberg integrals and hypergeometric functions associated with Macdonald symmetric polynomials*, in preparation.

[Z] D. Zeilberger, *A Stembridge-Stanton style elementary proof of the Habsieger-Kadell q-Morris identity*, Discrete Math. **79**(1989/90) 313-322.

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