A Fibonacci-Counting Proof Begged by Benjamin and Quinn

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Arthur Benjamin and Jennifer Quinn, in their delightful “Proofs that Really Count: The Art of Combinatorial Proof” (MAA, 2003), asked for a bijective proof of \((f_n := F_{n+1})\)

\[
\sum_i \binom{2n}{i} f_{2i-\varepsilon} = 5^n f_{2n-\varepsilon} \quad (\varepsilon \in \{0, 1\})
\]

Here goes. For any vector of integers \(u\), let \(|u|\) denote the sum of its entries. Let, for \(\varepsilon \in \{0, 1\}\),

\[
A_\varepsilon(n) := \{(w, u) : w \in \{0, 1\}^{2n}, \quad u \in \{1, 2\}^n, \quad 2|w| - |u| = \varepsilon \}
\]

\[
B_\varepsilon(n) := \{(w, u) : w \in \{1, 2, 3, 4, 5\}^n, \quad u \in \{1, 2\}^n, \quad |u| = 2n - \varepsilon \}
\]

The left side counts \(A_\varepsilon(n)\) and the right side counts \(B_\varepsilon(n)\). Both sets can be bijectively mapped to the set of \(n\)-step walks from \(A_0\) to \(A_0\) in a digraph whose vertices are \(A_0\) and \(A_1\), and there are ten edges from \(A_0\) to \(A_0\) and five edges each from \(A_0\) to \(A_1\), \(A_1\) to \(A_0\), \(A_1\) to \(A_1\).

For \((w, u) \in A_\varepsilon(n)\), write \(w = w'w''\), where \(\text{length}(w') = 2\), and \(u = u'u''\) where \(u'\) is the shortest head of \(u\) such that \(|u'| = 2|w'| - \varepsilon\) or \(|u'| = 2|w'| - \varepsilon + 1\), which entails \((w'', u'') \in A_0(n-1), (w'', u'') \in A_1(n-1)\), respectively. There are ten ways of getting from \(A_0(n)\) to \(A_0(n-1)\), five ways of getting from \(A_0(n)\) to \(A_1(n-1)\), five ways of getting from \(A_1(n)\) to \(A_0(n-1)\), and five ways of getting from \(A_1(n)\) to \(A_1(n-1)\). To get the walk, repeat this passage from \((w, u)\) to \((w'', u'')\), \(n\) times, recording the transitions, and the ‘states’.

For \((w, u) \in B_\varepsilon(n)\), write \(w = w'w''\), where \(\text{length}(w') = 1\), and \(u = u'u''\) where \(u'\) is the shortest head of \(u\) such that \(|u'| = 2 - \varepsilon\) or \(|u'| = 3 - \varepsilon\) which entails \((w'', u'') \in B_0(n-1), (w'', u'') \in B_1(n-1)\), respectively. There are ten ways of getting from \(B_0(n)\) to \(B_0(n-1)\), five ways of getting from \(B_0(n)\) to \(B_1(n-1)\), five ways of getting from \(B_1(n)\) to \(B_0(n-1)\), and five ways of getting from \(B_1(n)\) to \(B_1(n-1)\). To get the walk, repeat this passage from \((w, u)\) to \((w'', u'')\), \(n\) times, recording the transitions, and the ‘states’.

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1 \((w', u') = (00, 0), (01, 11), (01, 2), (10, 11), (10, 2), (11, 1111), (11, 112), (11, 121), (11, 211), (11, 22).
2 \((w', u') = (01, 12), (10, 12), (11, 1111), (11, 112), (11, 212).
3 \((w', u') = (01, 1), (10, 1), (11, 111), (11, 122), (11, 21).
4 \((w', u') = (00, 0), (01, 2), (10, 2), (11, 112), (11, 22).
1' \((w', u') = (1, 11), (2, 11), (3, 11), (4, 11), (5, 11), (1, 2), (2, 2), (3, 2), (4, 2), (5, 2).
2' \((w', u') = (1, 12), (2, 12), (3, 12), (4, 12), (5, 12).
3' \((w', u') = (1, 1), (2, 1), (3, 1), (4, 1), (5, 1).
4' \((w', u') = (1, 2), (2, 2), (3, 2), (4, 2), (5, 2).