FRACTIONAL COUNTING OF PARTITIONS

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1. Preamble

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, let us define $w_\lambda = \lambda_1 \lambda_2 \cdots \lambda_l$. The main quantity of interest is

$$b(n) = \sum_{\lambda \vdash n} \frac{1}{w_\lambda}.$$

We consider the generating function of this sequence:

$$f(z) = \sum_{n \geq 0} b(n) z^n = \prod_{i \geq 1} (1 - z^i/i)^{-1}.$$

Lemma 1.1. *The coefficients of*

$$(1-z)f(z) = \sum_{n \geq 0} (b(n) - b(n-1)) z^n = \prod_{i \geq 2} (1 - z^i/i)^{-1}.$$

are nonnegative.

Proof. If $\mu \cup \{1\}$ denotes the partition obtained from μ by adding a single part, we have that $w_{\mu \cup \{1\}} = w_\mu$, so

$$b(n) = \sum_{\lambda \vdash n} \frac{1}{w_\lambda} \geq \sum_{\mu \vdash n-1} \frac{1}{w_{\mu \cup \{1\}}} = b(n-1).$$

□

2. Asymptotics

Theorem 2.1. *We have $b(n) = e^{-\gamma} n(1 + o(1))$ as $n \to \infty$.*

Proof. Let us write, for $|z| < 1$:

$$\begin{aligned} \log(f(z)) &= -\log(1-z) - \sum_{i \geq 2} \log(1 - z^i/i) \\ &= -2\log(1-z) - z - \sum_{i \geq 2} (\log(1 - z^i/i) + z^i/i). \end{aligned}$$

By considering Taylor series, we have the estimate that

$$|\log(1-w) + w| < |w|^2$$

whenever $|w|$ is sufficiently small. If we define

$$g(z) = -z - \sum_{i \geq 2} (\log(1 - z^i/i) + z^i/i),$$

then for $|z| \leq 1$, all sufficiently large terms in the sum are bounded by $1/i^2$, in particular, $g(z)$ extends to a continuous function on the closed unit disc, and we have

$$\log(f(z)) = -2\log(1-z) + g(z)$$

which gives

$$(1-z)f(z) = \frac{e^{g(z)}}{1-z}$$

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We may now apply the Hardy-Littlewood Tauberian theorem. This theorem asserts that if $a_n \geq 0$ is a sequence of real numbers such that

$$\sum_{n\geq 0} a_n x^n \sim \frac{1}{1-x}$$

as $x \to 1$ from below, then

$$\sum_{k\leq n} a_k \sim n.$$

In our setting, Lemma 1 guarantees that we may apply the theorem after we multiply through by $e^{g(1)}$. We calculate

$$g(1) = -1 - \sum_{i\geq 2}(\log(1-1/i) + 1/i) = -\gamma$$

This proves the theorem. □

3. Understanding $b(n,k)$

In this section x will be a number between zero and one.

Definition 3.1. *Let*

$$c(n,k) = e^{\gamma} b(n, \lfloor k \rfloor)$$

and

$$c(n) = e^{\gamma} b(n).$$

Using this new function will make the following calculations cleaner, although it only negligibly differs from the function of interest. For example, $\lim_{n\to\infty} c(n)/n = 1$ according to our new convention. Note that $c(n,k)$ satisfies the same recurrence identities as $b(n, \lfloor k \rfloor)$.

Suppose that $1 \geq x \geq 1/2$. Then we have

$$c(n, xn) = \frac{1}{xn}\sum_{i=1}^{xn} c((1-x)n, i) = \frac{c((1-x)n)}{xn},$$

because $xn \geq (1-x)n$. By the result of the previous section, we may take the limit as $n \to \infty$, and obtain $\frac{1-x}{x}$.

Let us repeat this for $1/2 \geq x \geq 1/3$. We get

$$c(n, xn) = \frac{1}{xn}\sum_{i=1}^{xn} c((1-x)n, i) = \frac{1}{xn}\left(c((1-x)n) - \sum_{i=xn+1}^{(1-x)n} c((1-x)n, i)\right).$$

The key observation is that for i between $xn+1$ and $(1-x)n$, $2i \geq (1-x)n$, so that $c((1-x)n, i) = c((1-x)n - i)/i$ (similarly to the $1 \geq x \geq 1/2$ case). We may now take the limit as $n \to \infty$ (and recognise one of the terms as Riemann sum):

$$\begin{aligned}
\frac{1-x}{x} - \frac{1}{x}\lim_{n\to\infty}\sum_{i=xn+1}^{(1-x)n}\frac{(1-x)n - i + o(n)}{ni} &= \frac{1-x}{x} - \frac{1}{x}\lim_{n\to\infty}\frac{1}{n}\left(o(n) + \sum_{i=xn+1}^{(1-x)n}\frac{(1-x)-\frac{i}{n}}{\frac{i}{n}}\right) \\
&= \frac{1-x}{x} - \frac{1}{x}\int_x^{1-x}\frac{(1-x)-t}{t}dt \\
&= \frac{2-3x}{x} - \frac{(1-x)}{x}\log\left(\frac{1-x}{x}\right)
\end{aligned}$$

Here we used the fact that $i = \Theta(n)$, so that $\sum_{i=xn+1}^{(1-x)n} o(n)/i = o(n)$. We notice immediately that our limit function is not smooth at $x = 1/2$.

Proposition 3.2. *For $i \in \mathbb{Z}_{>0}$, there exists a smooth function $F_r(t)$ such that for $x \in [\frac{1}{r+1}, \frac{1}{r}]$,*

$$c(n, xn) = F_r(x) + o(1)$$

as $n \to \infty$.

Proof. We have already illustrated this in for $1 \geq x \geq 1/2$, where we obtained $F_1(x) = \frac{1-x}{x}$; this forms the base case of an induction on r. We now perform the same manipulation as for the case $1/2 \geq x \geq 1/3$, but instead assume $1/r \geq x \geq 1/(r+1)$ (which implies $1/(r-1) \geq \frac{x}{1-x} \geq 1/r$):

$$c(n, xn) = \frac{1}{xn} \sum_{i=1}^{xn} c((1-x)n, i) = \frac{1}{xn}\left(c((1-x)n) - \sum_{i=xn+1}^{(1-x)n} c((1-x)n, i) \right).$$

In this case, we use the fact that $(1-x)n \geq i \geq xn+1$ to deduce $1 \geq \frac{i}{(1-x)n} > \frac{x}{1-x} \geq 1/r$. We may therefore apply the induction hypothesis to the terms in the sum.

$$\begin{aligned} c(n, xn) &= \frac{1}{xn}\left(c((1-x)n) - \left(\sum_{i=xn+1}^{\frac{(1-x)n}{r-1}} F_{r-1}\left(\frac{i}{(1-x)n}\right) + o(1) \right.\right. \\ &= \left.\left. + \sum_{s=1}^{r-2} \sum_{i=\frac{(1-x)n}{(s+1)}+1}^{\frac{(1-x)n}{s}} F_s\left(\frac{i}{(1-x)n}\right) + o(1) \right)\right) \end{aligned}$$

Each term is a Riemann sum converging to an integral of the corresponding F_s. We note that although each $o(1)$ error term is summed $\mathcal{O}(n)$ times, this is accounted for by the leading factor of $1/n$, so these still vanish in the limit $n \to \infty$.

In particular, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{i=\frac{(1-x)n}{(s+1)}+1}^{\frac{(1-x)n}{s}} F_s\left(\frac{i}{(1-x)n}\right) &= \int_{\frac{1-x}{s+1}}^{\frac{1-x}{s}} F_s\left(\frac{t}{1-x}\right) dt \\ &= (1-x) \int_{\frac{1}{s+1}}^{\frac{1}{s}} F_s(t) dt. \end{aligned}$$

We conclude that

$$\lim_{n\to\infty} c(n, xn) = \frac{1-x}{x} - \frac{1-x}{x}\left(\int_{\frac{x}{1-x}}^{\frac{1}{r-1}} F_{r-1}(t)dt + \sum_{s=1}^{r-2} \int_{\frac{1}{s+1}}^{\frac{1}{s}} F_s(t)dt \right).$$

For $x \in [1/(r+1), 1/r]$, it is this quantity which we define to be $F_r(x)$, and the above limit is exactly the statement of the proposition. We conclude that $\lim_{n\to\infty} c(n, nx)$ is continuous, but fails to be smooth at $\{1/n \mid n \in \mathbb{Z}_{>0}\}$. ◻

We may differentiate the integral definition of $F_r(x)$ to see what differential equation it can satisfy.

$$\frac{d}{dx}\left(\frac{x}{1-x} F_r(x)\right) = \frac{1}{(1-x)^2} F_{r-1}\left(\frac{x}{1-x}\right)$$

This rearranges to

$$F_r(x) + x(1-x)F_r'(x) = F_{r-1}\left(\frac{x}{1-x}\right). \tag{1}$$

Corollary 3.3. *Because $c(n,k)$ and $b(n,k)$ differed only by rescaling, and the above relations are linear in the F_r, we have*

$$\lim_{n\to\infty} b(n,xn) = e^{-\gamma}F_r(x)$$

whenever $1/r \geq x \geq 1/(r+1)$.

4. Another approach

There is another path to finding the value $e^{-\gamma}$. As soon as one knows that the limit $\lim_{n\to\infty} b(n)/n$ exists, one may determine the value of the limit as follows. The methods of Section 3 do not require the limit to be known, so we have access to Equation 1 (the differential equation satisfied by F_r).

Remark 4.1. *The differential equation can be guessed by taking the formula*

$$b(n,k) = \frac{k-1}{k}b(n-1,k-1) + \frac{1}{k}b(n-k,k),$$

setting $k = xn$, substituting the ansatz $F(x) = b(n,xn)$, and rearranging to obtain

$$\frac{F(x) - F\left(x - \frac{1-x}{n-1}\right)}{\frac{1-x}{n-1}} = \frac{\frac{1}{xn}\left(F\left(\frac{x}{1-x}\right) - F\left(x - \frac{1-x}{n-1}\right)\right)}{\frac{1-x}{n-1}},$$

and taking the limit $n \to \infty$ (where the left hand side becomes $F'(x)$).

Let $y = 1/x$ and $G(y) = F(1/x)$ (where we unite all F_r into a single function defined for $0 \leq x \leq 1$). Then, the differential equation becomes

$$G(y) - (y-1)G'(y) = G(y-1)$$

which transforms the intervals $1/r \geq x \geq 1/(r+1)$ into $r \leq y \leq r+1$. The upshot of this is that the current equation is well adapted for a Laplace transform. Writing $\hat{G}(t)$ for the Laplace transform of G, we obtain:

$$\hat{G}(t) + (t\hat{G}(t) - G(0)) + \frac{d}{dt}(t\hat{G}(t) - G(0)) = e^{-t}\hat{G}(t),$$

using the boundary condition $G(0) = 0$, this becomes

$$\frac{d}{dt}\hat{G}(t) = \frac{e^{-t} - t - 2}{t}\hat{G}(t).$$

We may solve this explicitly, and $\log(\hat{G}(t))$ turns out to be very similar to an exponential integral. We may now use standard properties of Laplace transforms to show express

$$\frac{1}{G(\infty)}\int_1^\infty G(y)\frac{dy}{y^2} = \frac{1}{F(0)}\int_0^1 F(x)dx$$

in terms of integrals of the exponential integral or logarithmic integral (depending on how it is set up); these can be solved exactly to obtain $e^{-\gamma}$. Thus the problem reduces to showing that $\lim_{x\to 0} F(x) = 1$. It is not difficult to show that $b(n,k) \leq 1$ by induction on k, so the problem reduces to finding an appropriate lower bound for $b(n,k)$; I believe this can be done by induction also, but the argument is more elaborate.