AN ANSWER TO A QUESTION OF ZEILBERGER AND ZEILBERGER ABOUT FRACTIONAL COUNTING OF PARTITIONS

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Abstract. We answer a question of Zeilberger and Zeilberger about certain partition statistics.

1. Introduction

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, define $w_\lambda = \lambda_1 \lambda_2 \cdots \lambda_l$ (this is the product of the parts of λ). Zeilberger and Zeilberger [ZZ18] define two quantities:

$$b(n) = \sum_{\lambda \vdash n} \frac{1}{w_\lambda}.$$

and

$$b(n,k) = \sum_{\substack{\lambda \vdash n \\ \lambda_1 = k}} \frac{1}{w_\lambda}.$$

The latter sum is over partitions of n whose largest part is equal to k, so $b(n) = \sum_{i=1}^n b(n,k)$. They ask to determine

$$f(x) = \lim_{n\to\infty} b(n, \lfloor xn \rfloor)$$

as a function on $[0,1]$. To answer this question, we use two tools. Firstly, a recurrence for $b(n,k)$ given by Zeilberger and Zeilberger [ZZ18]:

$$b(n,k) = \frac{1}{k} \sum_{i=1}^{k} b(n-k, i).$$

Secondly, we use the asymptotic behaviour of $b(n)$, first considered by Lehmer [Leh72].

Theorem 1.1 (Lehmer). *We have $b(n) = e^{-\gamma} n(1 + o(1))$ as $n \to \infty$, where γ is Euler's gamma.*

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2. Understanding $b(n,k)$

In this section x will be a number in $[0,1]$.

Definition 2.1. *Let*

$$c(n,k) = e^{\gamma} b(n,k)$$

and

$$c(n) = e^{\gamma} b(n).$$

Using this new function will make the following calculations cleaner. For example, $\lim_{n\to\infty} c(n)/n = 1$ according to our new convention. Note that $c(n,k)$ satisfies the same recurrence identities as $b(n,k)$.

Example 2.2. *Suppose that $x \in (1/2, 1]$. Then for n sufficiently large, we have*

$$c(n, \lfloor xn \rfloor) = \frac{1}{\lfloor xn \rfloor} \sum_{i=1}^{\lfloor xn \rfloor} c(n - \lfloor xn \rfloor, i) = \frac{c(n - \lfloor xn \rfloor)}{\lfloor xn \rfloor},$$

because $\lfloor xn \rfloor \geq n - \lfloor xn \rfloor$ for n sufficiently large. By Theorem 1.1, we may take the limit as $n \to \infty$, and obtain $\frac{1-x}{x}$.

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Proposition 2.3. *For $r \in \mathbb{Z}_{>0}$, there exists a smooth function $F_r(t)$ such that for $x \in (\frac{1}{r+1}, \frac{1}{r}]$,*

$$c(n, \lfloor xn \rfloor) = F_r(x) + o(1)$$

as $n \to \infty$. Moreover, these $F_r(x)$ are related via

$$F_r(x) = \frac{1-x}{x} - \frac{1-x}{x}\left(\int_{\frac{x}{1-x}}^{\frac{1}{r-1}} F_{r-1}(t)dt + \sum_{s=1}^{r-2}\int_{\frac{1}{s+1}}^{\frac{1}{s}} F_s(t)dt\right).$$

Proof. Example 2.2 demonstrated this for $x \in (1/2, 1]$, where we obtained $F_1(x) = \frac{1-x}{x}$; this forms the base case of an induction on r. We now assume $x \in (\frac{1}{r+1}, \frac{1}{r}]$;

$$c(n, \lfloor xn \rfloor) = \frac{1}{\lfloor xn \rfloor}\sum_{i=1}^{\lfloor xn \rfloor} c(n - \lfloor xn \rfloor, i) = \frac{1}{\lfloor xn \rfloor}\left(c(n - \lfloor xn \rfloor) - \sum_{i=\lfloor xn \rfloor + 1}^{n - \lfloor xn \rfloor} c(n - \lfloor xn \rfloor, i)\right).$$

In the latter sum, the ratio $\frac{i}{n - \lfloor xn \rfloor}$ is minimised when $i = \lfloor xn \rfloor + 1$, and the resulting quantity is a weakly decreasing function of x. Because $x > \frac{1}{r+1}$, we conclude

$$\frac{i}{n - \lfloor xn \rfloor} \geq \frac{\lfloor \frac{n}{r+1} \rfloor + 1}{n - \lfloor \frac{n}{r+1} \rfloor} \geq 1/r.$$

We may therefore apply the induction hypothesis to the terms in the sum.

$$\begin{aligned} c(n, \lfloor xn \rfloor) &= \frac{1}{\lfloor xn \rfloor}\left(c(n - \lfloor xn \rfloor) - \left(\sum_{i=\lfloor xn \rfloor + 1}^{\lfloor \frac{n - \lfloor xn \rfloor}{r-1} \rfloor} F_{r-1}\left(\frac{i}{n - \lfloor xn \rfloor}\right) + o(1)\right.\right. \\ &= \left.\left. + \sum_{s=1}^{r-2}\sum_{i=\lfloor \frac{n - \lfloor xn \rfloor}{(s+1)} \rfloor + 1}^{\lfloor \frac{n - \lfloor xn \rfloor}{s} \rfloor} F_s\left(\frac{i}{n - \lfloor xn \rfloor}\right) + o(1)\right)\right) \end{aligned}$$

Each term is a Riemann sum converging to an integral of the corresponding F_s. We note that although each $o(1)$ error term is summed $\mathcal{O}(n)$ times, this is accounted for by the leading factor of $1/\lfloor xn \rfloor$, so these still vanish in the limit $n \to \infty$. Note that we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=\lfloor \frac{n - \lfloor xn \rfloor}{(s+1)} \rfloor + 1}^{\lfloor \frac{n - \lfloor xn \rfloor}{s} \rfloor} F_s\left(\frac{i}{n - \lfloor xn \rfloor}\right) = \int_{\frac{1-x}{s+1}}^{\frac{1-x}{s}} F_s\left(\frac{t}{1-x}\right)dt = (1-x)\int_{\frac{1}{s+1}}^{\frac{1}{s}} F_s(t)dt.$$

We conclude that

$$\lim_{n\to\infty} c(n, \lfloor xn \rfloor) = \frac{1-x}{x} - \frac{1-x}{x}\left(\int_{\frac{x}{1-x}}^{\frac{1}{r-1}} F_{r-1}(t)dt + \sum_{s=1}^{r-2}\int_{\frac{1}{s+1}}^{\frac{1}{s}} F_s(t)dt\right).$$

For $x \in (\frac{1}{r+1}, \frac{1}{r}]$, it is this quantity which we define to be $F_r(x)$, and the above limit is exactly the statement of the proposition. We conclude that $\lim_{n\to\infty} c(n, \lfloor nx \rfloor)$ is smooth for $x \notin \{1/n \mid n \in \mathbb{Z}_{>0}\}$. □

Example 2.4. *We may compute*

$$F_2(x) = \frac{1-x}{x} - \frac{1-x}{x}\left(\int_{\frac{x}{1-x}}^{1} \frac{1-t}{t}dt\right) = \frac{2-3x}{x} - \frac{1-x}{x}\log\left(\frac{1-x}{x}\right).$$

Remark 2.5. *We may differentiate the expression for $F_r(x)$ to obtain a differential equation satisfied by $F_r(x)$:*

$$\frac{d}{dx}\left(\frac{x}{1-x}F_r(x)\right) = \frac{1}{(1-x)^2}F_{r-1}\left(\frac{x}{1-x}\right)$$

Finally, we obtain our result.

Corollary 2.6. *Because $c(n,k)$ and $b(n,k)$ differed only by rescaling, and the above relations are linear in the F_r, we have*

$$\lim_{n\to\infty} b(n, \lfloor xn \rfloor) = e^{-\gamma} F_r(x)$$

whenever $x \in (\frac{1}{r+1}, \frac{1}{r}]$.

Remark 2.7. *Suppose we assemble all the functions $F_r(x)$ into a single function $F(x)$ on $(0,1]$. Let $G(x) = F(1/x)$. Then, the differential equation becomes*

$$G(x) - (x-1)G'(x) = G(x-1)$$

The upshot of this is that the current equation is well adapted for a Laplace transform. Writing $\hat{G}(t)$ for the Laplace transform of $G(x)$, we obtain:

$$\hat{G}(t) + (t\hat{G}(t) - G(0)) + \frac{d}{dt}(t\hat{G}(t) - G(0)) = e^{-t}\hat{G}(t),$$

using the boundary condition $G(0) = 0$, this becomes

$$\frac{d}{dt}\hat{G}(t) = \frac{e^{-t} - t - 2}{t}\hat{G}(t).$$

We may solve this explicitly:

$$\hat{G}(t) = Kt^{-2}\exp(Ei(-t) - t),$$

where Ei is the exponential integral, and K is a constant.

References

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