# A Surprising Identity between two integrals [Temporary Title]

*AuthorsHidden*

**Proposition**: For real $a > b > 0$ and non-negative integer $n$, the following beautiful and surprising identity holds.

$$\int_0^1 \frac{x^n\,(1-x)^n}{((x+a)(x+b))^{n+1}}\,dx \;=\; \int_0^1 \frac{x^n\,(1-x)^n}{((a-b)x+(a+1)b)^{n+1}}\,dx \quad .$$

**Proof**: Fix $a$ and $b$, and let $L(n)$ and $R(n)$ be the *integrals* on the left and right sides respectively, and let $F_1(n,x)$, and $F_2(n,x)$ be the corresponding *integrands*, so that $L(n) = \int_0^1 F_1(n,x)\,dx$ and $R(n) = \int_0^1 F_2(n,x)\,dx$. We cleverly construct the rational functions

$$R_1(x) \;=\; \frac{x\,(x-1)\left((a+b+1)x^2+2abx-ab\right)}{(x+b)\,(x+a)}\;,\; R_2(x) \;=\; \frac{x\,(x-1)\left((a-b)x^2+2b(a+1)x+(a+1)b\right)}{(a-b)x+(a+1)b}\;,$$

with the motive that (check!)

$$(n+1)\,F_1(n,x)-(2\,n+3)\,(2\,ba+a+b)\,F_1\,(n+1,x)+(a-b)^2\,(n+2)\,F_1\,(n+2,x) \;=\; \frac{d}{dx}(R_1(x)F_1(n,x)) \quad ,$$

$$(n+1)\,F_2(n,x)-(2\,n+3)\,(2\,ba+a+b)\,F_2\,(n+1,x)+(a-b)^2\,(n+2)\,F_2\,(n+2,x) \;=\; \frac{d}{dx}(R_2(x)F_2(n,x)) \quad .$$

Integrating both identities from $x=0$ to $x=1$, and noting that the right sides vanish, we have

$$(n+1)\,L\,(n)-(2\,n+3)\,(2\,ba+a+b)\,L\,(n+1)+(a-b)^2\,(n+2)\,L\,(n+2) \;=\; 0 \quad ,$$

$$(n+1)\,R\,(n)-(2\,n+3)\,(2\,ba+a+b)\,R\,(n+1)+(a-b)^2\,(n+2)\,R\,(n+2) \;=\; 0 \quad .$$

Since $L(0)=R(0)$ and $L(1)=R(1)$ (check!), the proposition follows by mathematical induction.

**Comments**:**1**. This beatiful identity is equivalent to an identity buried in Bailey's classic book [B], section 9.5, formula (2), but you need an expert (like the third-named author) to realize that! **2.** Our proof was obtained by the first named-author, running a Maple program, `HIDDEN`, written by the second-named author, that implements the Almkvist-Zeilberger algorithm [AZ] designed by Hidden and our good mutual friend Hidden, to whose memory this note is dedicated. **3.** The integrals are not taken from a pool of no-one-cares analytic creatures: the right-hand side covers a famous sequence of rational approximations to $\log(1+(a-b)/((a+1)b))$ [AR]. Hence the left-hand side does.

## References


[AR] K. Alladi and M. L. Robertson, *Legendre polynomials and irrationality*, J. Reine Angew. Math. **318**(1980), 137-155.

[AZ] Gert Almkvist and Doron Zeilberger, *The method of differentiating under the integral sign*, J. Symbolic Computation **10**(1990), 571-591;
`http://www.math.rutgers.edu/~zeilberg/mamarimY/duis.pdf`

[B] W. N. Bailey, *"Generalized hypergeometric series"*, Cambridge University Press, 1935.

---

HIDDEN

---

Written: Nov.3, 2019.