Two Definite Integrals That Are Definitely (and Surprisingly!) Equal

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To our good friend Gert Almkvist (1934-2018), In Memoriam

Proposition: For real $a > b > 0$ and non-negative integer n, the following beautiful and surprising identity holds.

$$\int_0^1 \frac{x^n\,(1-x)^n}{((x+a)(x+b))^{n+1}}\,dx \;=\; \int_0^1 \frac{x^n\,(1-x)^n}{((a-b)x+(a+1)b)^{n+1}}\,dx \quad .$$

Proof: Fix a and b, and let $L(n)$ and $R(n)$ be the *integrals* on the left and right sides respectively, and let $F_1(n,x)$, and $F_2(n,x)$ be the corresponding *integrands*, so that $L(n) = \int_0^1 F_1(n,x)\,dx$ and $R(n) = \int_0^1 F_2(n,x)\,dx$. We cleverly construct the rational functions

$$R_1(x) \;=\; \frac{x\,(x-1)\left((a+b+1)x^2+2abx-ab\right)}{(x+b)\,(x+a)} \;,\; R_2(x) \;=\; \frac{x\,(x-1)\left((a-b)x^2+2b(a+1)x-(a+1)b\right)}{(a-b)x-(a+1)b} \;,$$

with the motive that (check!)

$$(n+1)\,F_1(n,x) - (2\,n+3)\,(2\,ba+a+b)\,F_1\,(n+1,x) + (a-b)^2\,(n+2)\,F_1\,(n+2,x) \;=\; \frac{d}{dx}(R_1(x)F_1(n,x)) \quad ,$$

$$(n+1)\,F_2(n,x) - (2\,n+3)\,(2\,ba+a+b)\,F_2\,(n+1,x) + (a-b)^2\,(n+2)\,F_2\,(n+2,x) \;=\; \frac{d}{dx}(R_2(x)F_2(n,x)) \quad .$$

Integrating both identities from $x = 0$ to $x = 1$, and noting that the right sides vanish, we have

$$(n+1)\,L\,(n) - (2\,n+3)\,(2\,ba+a+b)\,L\,(n+1) + (a-b)^2\,(n+2)\,L\,(n+2) \;=\; 0 \quad ,$$

$$(n+1)\,R\,(n) - (2\,n+3)\,(2\,ba+a+b)\,R\,(n+1) + (a-b)^2\,(n+2)\,R\,(n+2) \;=\; 0 \quad .$$

Since $L(0) = R(0)$ and $L(1) = R(1)$ (check!), the proposition follows by mathematical induction.

Comments:**1**. This beatiful identity is equivalent to an identity buried in Bailey's classic book [B], section 9.5, formula (2), but you need an expert (like the third-named author) to realize that! **2.** Our proof was obtained by the first named-author, running a Maple program, `http://sites.math.rutgers.edu/~zeilberg/tokhniot/EKHAD.txt` , written by the second-named author, that implements the Almkvist-Zeilberger algorithm [AZ] designed by Zeilberger and our good mutual friend Gert Almkvist, to whose memory this note is dedicated. **3.** The integrals are not taken from a pool of no-one-cares-about analytic creatures: the right-hand side covers a famous sequence of rational approximations to $\log(1+(a-b)/((a+1)b))$ [AR], hence the left-hand side does too. **4.** We thank Greg Egan for spotting a sign error in an earlier version. **5.** Watch Greg Egan's beautiful animation in `https://twitter.com/gregeganSF/status/1192309179119104000`. **6.** To our surprise, the identity is not as suprising as we believed. Mikael Sundquist noticed that the change of variable $x = b(1-u)/(b+u)$ gives a 'calc1 proof'. Indeed $dx = -\frac{b(1+b)}{(b+u)^2}\,du$, and we have

$$\int_0^1 \frac{x^n\,(1-x)^n}{((x+a)(x+b))^{n+1}}\,dx \;=\; \int_0^1 \frac{(b(1-u)/(b+u))^n\,(1-(b(1-u)/(b+u)))^n}{(b(1-u)/(b+u)+a)(b(1-u)/(b+u)+b))^{n+1}}\,\frac{b(1+b)}{(b+u)^2}\,du$$

$$= \int_0^1 \frac{(1-u)^n\, u^n\, b^{n+1}(1+b)^{n+1}}{(\, b(1-u)+a(b+u))(b(1-u)+b(b+u))\,)^{n+1}}\, du = \int_0^1 \frac{u^n(1-u)^n}{((a-b)u+(a+1)b)^{n+1}}\, du \quad .$$

Note that n does not have to be an integer.

7. Alin Bostan has two further insightful proofs. See
`https://specfun.inria.fr/bostan/publications/EZZ.pdf` and
`https://specfun.inria.fr/bostan/publications/EZZ2.pdf`.

References

[AR] K. Alladi and M. L. Robinson, *Legendre polynomials and irrationality*, J. Reine Angew. Math. **318**(1980), 137-155.

[AZ] Gert Almkvist and Doron Zeilberger, *The method of differentiating under the integral sign*, J. Symbolic Computation **10**(1990), 571-591;
`http://www.math.rutgers.edu/~zeilberg/mamarimY/duis.pdf`

[B] W. N. Bailey, *"Generalized hypergeometric series"*, Cambridge University Press, 1935.

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