

## Two Definite Integrals That Are Definitely (and Surprisingly!) Equal [Second Edition]

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To our good friend Gert Almkvist (1934-2018), In Memoriam

**Proposition:** For real  $a > b > 0$  and non-negative integer  $n$ , the following beautiful and surprising identity holds.

$$\int_0^1 \frac{x^n (1-x)^n}{((x+a)(x+b))^{n+1}} dx = \int_0^1 \frac{x^n (1-x)^n}{((a-b)x + (a+1)b)^{n+1}} dx \quad .$$

**Proof:** Fix  $a$  and  $b$ , and let  $L(n)$  and  $R(n)$  be the *integrals* on the left and right sides respectively, and let  $F_1(n, x)$ , and  $F_2(n, x)$  be the corresponding *integrands*, so that  $L(n) = \int_0^1 F_1(n, x) dx$  and  $R(n) = \int_0^1 F_2(n, x) dx$ . We cleverly construct the rational functions

$$R_1(x) = \frac{x(x-1)((a+b+1)x^2 + 2abx - ab)}{(x+b)(x+a)}, \quad R_2(x) = \frac{x(x-1)((a-b)x^2 + 2b(a+1)x - (a+1)b)}{(a-b)x - (a+1)b},$$

with the motive that (check!)

$$(n+1)F_1(n, x) - (2n+3)(2ba+a+b)F_1(n+1, x) + (a-b)^2(n+2)F_1(n+2, x) = \frac{d}{dx}(R_1(x)F_1(n, x)) \quad ,$$

$$(n+1)F_2(n, x) - (2n+3)(2ba+a+b)F_2(n+1, x) + (a-b)^2(n+2)F_2(n+2, x) = \frac{d}{dx}(R_2(x)F_2(n, x)) \quad .$$

Integrating both identities from  $x = 0$  to  $x = 1$ , and noting that the right sides vanish, we have

$$(n+1)L(n) - (2n+3)(2ba+a+b)L(n+1) + (a-b)^2(n+2)L(n+2) = 0 \quad ,$$

$$(n+1)R(n) - (2n+3)(2ba+a+b)R(n+1) + (a-b)^2(n+2)R(n+2) = 0 \quad .$$

Since  $L(0) = R(0)$  and  $L(1) = R(1)$  (check!), the proposition follows by mathematical induction.

**Comments:1.** This beautiful identity is equivalent to an identity buried in Bailey's classic book [B], section 9.5, formula (2), but you need an expert (like the third-named author) to realize that!

**2.** Our proof was obtained by the first named-author, running a Maple program,

<http://sites.math.rutgers.edu/~zeilberg/tokhniot/EKHAD.txt>, written by the second-named

author, that implements the Almkvist-Zeilberger algorithm [AZ] designed by Zeilberger and our good mutual friend Gert Almkvist, to whose memory this note is dedicated. **3.** The integrals are not taken from a pool of no-one-cares analytic creatures: the right-hand side covers a famous sequence of rational approximations to  $\log(1 + (a-b)/((a+1)b))$  [AR]. Hence the left-hand side does.

### Additional Comments (added in 2nd edition)

**4.** This version corrects a sign typo pointed out by Greg Egan. **5.** Watch Greg Egan's beautiful animation in <https://twitter.com/gregeganSF/status/1192309179119104000>. **6.** To

our surprise, the identity is not as suprising as we believed. Mikael Sundquist noticed that the change of variable  $x = b(1 - u)/(b + u)$  gives a ‘calc1 proof’. **7.** Alin Bostan has two further insightful proofs. See <https://specfun.inria.fr/bostan/publications/EZZ.pdf> and <https://specfun.inria.fr/bostan/publications/EZZ2.pdf>.

## References

[AR] K. Alladi and M. L. Robertson, *Legendre polynomials and irrationality*, J. Reine Angew. Math. **318**(1980), 137-155.

[AZ] Gert Almkvist and Doron Zeilberger, *The method of differentiating under the integral sign*, J. Symbolic Computation **10**(1990), 571-591;  
<http://www.math.rutgers.edu/~zeilberg/mamarimY/duis.pdf>

[B] W. N. Bailey, “*Generalized hypergeometric series*”, Cambridge University Press, 1935.

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