Two Definite Integrals That Are Definitely (and Surprisingly!) Equal

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To our good friend Gert Almkvist (1934-2018), In Memoriam

Proposition: For real $a > b > 0$ and non-negative integer n, the following beautiful and surprising identity holds.

$$\int_0^1 \frac{x^n\,(1-x)^n}{((x+a)(x+b))^{n+1}}\,dx \;=\; \int_0^1 \frac{x^n\,(1-x)^n}{((a-b)x+(a+1)b)^{n+1}}\,dx \quad .$$

Proof: Fix a and b, and let $L(n)$ and $R(n)$ be the *integrals* on the left and right sides respectively, and let $F_1(n,x)$, and $F_2(n,x)$ be the corresponding *integrands*, so that $L(n) = \int_0^1 F_1(n,x)\,dx$ and $R(n) = \int_0^1 F_2(n,x)\,dx$. We cleverly construct the rational functions

$$R_1(x) \;=\; \frac{x\,(-1+x)\left(2\,abx+ax^2+bx^2-ba+x^2\right)}{(x+b)\,(x+a)}\,,\; R_2(x) \;=\; \frac{x\,(-1+x)\left(2\,abx+ax^2-bx^2-ba+2\,xb-b\right)}{ba+xa-xb+b}\,,$$

with the motive that (check!)

$$(n+1)\,F_1(n,x) - (2\,n+3)\,(2\,ba+a+b)\,F_1\,(n+1,x) + (a-b)^2\,(n+2)\,F_1\,(n+2,x) \;=\; \frac{d}{dx}(R_1(x)F_1(n,x)) \quad ,$$

and

$$(n+1)\,F_2(n,x) - (2\,n+3)\,(2\,ba+a+b)\,F_2\,(n+1,x) + (a-b)^2\,(n+2)\,F_2\,(n+2,x) \;=\; \frac{d}{dx}(R_2(x)F_2(n,x)) \quad .$$

Integrating both identities from $x = 0$ to $x = 1$, and noting that the right sides vanish, we have

$$(n+1)\,L\,(n) - (2\,n+3)\,(2\,ba+a+b)\,L\,(n+1) + (a-b)^2\,(n+2)\,L\,(n+2) \;=\; 0 \quad ,$$

$$(n+1)\,R\,(n) - (2\,n+3)\,(2\,ba+a+b)\,R\,(n+1) + (a-b)^2\,(n+2)\,R\,(n+2) \;=\; 0 \quad .$$

Since $L(0) = R(0)$ and $L(1) = R(1)$ (check!), the proposition follows by mathematical induction. $\square$

Comments;

1. This beatiful identity is equivalent to an identity buried in Bailey's classic book [B], section 9.5, formula (2), but you need an expert (like the third-named author) to realize that!

2. Our proof was obtained by the first named-author, running a Maple program, `http://sites.math.rutgers.edu/~zeilberg/tokhniot/EKHAD.txt` , written by the second-named author, that implements the Almkvist-Zeilberger algorithm [AZ] designed by Zeilberger and our good mutual friend Gert Almkvist, to whose memory this note is dedicated.

References

[AZ] Gert Almkvist and Doron Zeilberger, *The method of differentiating under the integral sign*, J. Symbolic Computation **10**(1990), 571-591;
`http://www.math.rutgers.edu/~zeilberg/mamarimY/duis.pdf`

[B] W. N. Bailey, *"Generalized hypergeometric series"*, Cambridge University Press, 1935.

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