A Symbolic Computation Approach to a Problem involving Multivariate Poission Distributions

(Draft, do not distribute!)

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Introduction

Eduardo, please write it.

The Problem

Suppose that we have n independent Poisson random variables, X_i ($i = 1 \dots n$) with parameters λ_i respectively. In other words

$$Pr(X_1 = k_1, \dots, X_n = k_n) = \frac{\lambda_1^{k_1}}{k_1!} \frac{\lambda_2^{k_2}}{k_2!} \dots \frac{\lambda_n^{k_n}}{k_n!}$$

Suppose that we can't observe the X_i's directly, but only a certain number, m, of linear combinations of them:

$$Y_j = \sum_{i=1}^{n} a_{ij} X_j \quad , \quad (j = 1, \dots, m) \quad ,$$

where (a_{ij}) is a certain $m \times n$ matrix with non-negative coefficients.

We are interested in the following questions:

1. Can one compute (fast!), for any given vector $(b_1, \dots, b_m)$ (possibly with large coordinates), the probability

$$F(b_1, \dots, b_m) := Pr(Y_1 = b_1, \dots, Y_m = b_m)$$

2. Can one compute (fast!), for any given vectors $(b_1, \dots, b_m)$, (possibly with large coordinates) the conditional expectation

$$G_i(b_1, \dots, b_m) := E[X_i \, \Big| \, Y_1 = b_1, \dots, Y_m = b_m]$$

3. More generally, can one compute (fast!), the higher moments

$$G_i^{(r)}(b_1, \dots, b_m) := E[X_i^r \, \Big| \, Y_1 = b_1, \dots, Y_m = b_m], (r \geq 2) \quad .$$

(which would immediately allow us to compute the moments about the mean.)

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The generating function

Fix a matrix $A = (a_{ij})$ $(1 \le i \le m,\ 1 \le j \le n)$. Let $f(z_1, \ldots, z_m)$ be the (multivariable) generating function of $F(b_1, \ldots, b_m)$, in other words

$$f(z_1, \ldots, z_m) = \sum_{b_1 \ge 0, \ldots, b_m \ge 0} F(b_1, \ldots, b_m) z_1^{b_1} \cdots z_m^{b_m} \quad .$$

We have:

$$f(z_1, \ldots, z_m) = \sum_{b_1 \ge 0, \ldots, b_m \ge 0} F(b_1, \ldots, b_m) z_1^{b_1} \cdots z_m^{b_m} =$$

$$= \sum_{b_1 \ge 0, \ldots, b_m \ge 0} \left(\sum_{\substack{k_1, \ldots, k_n \ge 0 \\ a_{11}k_1 + \ldots + a_{1n}k_n = b_1, \ldots, a_{m1}k_1 + \ldots + a_{mn}k_n = b_m}} \frac{\lambda_1^{k_1}}{k_1!} \frac{\lambda_2^{k_2}}{k_2!} \cdots \frac{\lambda_n^{k_n}}{k_n!} \right) z_1^{b_1} \cdots z_m^{b_m}$$

By changing the order of summation, this equals

$$\sum_{k_1 \ge 0, \ldots, k_n \ge 0} \frac{\lambda_1^{k_1}}{k_1!} \frac{\lambda_2^{k_2}}{k_2!} \cdots \frac{\lambda_n^{k_n}}{k_n!} z_1^{a_{11}k_1 + \ldots + a_{1n}k_n} \cdots z_m^{a_{m1}k_1 + \ldots + a_{mn}k_n}$$

$$= \sum_{k_1 \ge 0, \ldots, k_n \ge 0} \frac{(\lambda_1 z_1^{a_{11}} z_2^{a_{21}} \cdots z_m^{a_{m1}})^{k_1}}{k_1!} \cdots \frac{(\lambda_n z_1^{a_{1n}} z_2^{a_{2n}} \cdots z_m^{a_{mn}})^{k_n}}{k_n!}$$

$$= \left(\sum_{k_1 \ge 0} \frac{(\lambda_1 z_1^{a_{11}} z_2^{a_{21}} \cdots z_m^{a_{m1}})^{k_1}}{k_1!} \right) \cdots \left(\sum_{k_n \ge 0} \frac{(\lambda_n z_1^{a_{1n}} z_2^{a_{2n}} \cdots z_m^{a_{mn}})^{k_n}}{k_n!} \right)$$

$$= exp(\lambda_1 z_1^{a_{11}} z_2^{a_{21}} \cdots z_m^{a_{m1}}) \cdots exp(\lambda_n z_1^{a_{1n}} z_2^{a_{2n}} \cdots z_m^{a_{mn}})$$

$$= exp\left(\lambda_1 z_1^{a_{11}} z_2^{a_{21}} \cdots z_m^{a_{m1}} + \cdots + \lambda_n z_1^{a_{1n}} z_2^{a_{21}} \cdots z_m^{a_{mn}}\right) \quad .$$

We have just derived

Theorem 1: The generating function $f(z_1, \ldots, z_m)$ is given by

$$f(z) = exp\left(\sum_j^n \lambda_j \prod_{i=1}^m z_i^{a_{ij}} \right) \quad .$$

So $P(b_1, \ldots, b_m)$ is the coefficient of $z_1^{b_1} \cdots z_m^{b_m}$ in the multivariable Taylor expansion about the origin of $f(z_1, \ldots, z_m)$.

To get the factorial moments of X_i, $E[X_i^{(r)}]$ we apply the operator $\lambda_i^r (\frac{\partial}{\partial \lambda_i})^r$ to $P(b_1, \ldots, b_m)$ and divide by $P(b_1, \ldots, b_m)$. By doing the same for the generating function, we see that the numerator is λ_i^r times the coefficient of $(\prod_{i=1}^m z_i^{a_{ij}})^r$, In other words we have

Theorem 2: The factorial moments $E[X_j^{(r)}]$ are given in terms of the $P(b_1, \ldots, b_m)$ by

$$\frac{P(b_1 - ra_{1j}, b_2 - ra_{2j}, \ldots, b_m - ra_{mj})}{P(b_1, \ldots, b_m)}$$

So everything depends on a fast computation of the coefficients of $f(z_1, \ldots, z_m)$.