Babson-Steingrímsson Statistics Are Indeed Mahonian (and Sometimes Even Euler-Mahonian)

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Abstract: Babson and Steingrímsson have recently introduced eight new permutation statistics, that they conjectured were all Mahonian (i.e. equi-distributed with the number of inversions). We prove their conjecture for the first four, and also prove that the first and the fourth are even Euler-Mahonian. We use two different, in fact, opposite, techniques. For three of them we give a computer-generated proof, using the Maple package `ROTA`, that implements the second author's "Umbral Transfer Matrix Method." For the fourth one a geometric permutation transformation is used that leads to a further refinement of this Euler-Mahonian distribution study.

1. Babson and Steingrímsson's Notation

In [BaSt00] Babson and Steingrímsson introduced a convenient notation for "atomic" permutation statistics. Given a permutation $w = x_1x_2\dots x_n$ of $1,2,\dots,n$ they define, for example, $(a\text{-}bc)(w)$ to be the number of occurrences of the "pattern" $a\text{-}bc$, i.e. the number of pairs of places $1 \le i < j < n$ such that $x_i < x_j < x_{j+1}$. Similarly, the pattern $(b\text{-}ca)(w)$ is that number of occurrences of $x_{j+1} < x_i < x_j$, and in general, for any permutation α,β,γ of a,b,c, the expression $(\alpha\text{-}\beta\gamma)(w)$ is the number of pairs (i,j), $1 \le i < j < n$, such that the orderings of the two triples (x_i, x_j, x_{j+1}) and (α,β,γ) are identical. The statistic $(ab\text{-}c)(w)$ is defined in the same way by looking at the occurrences (x_i, x_{i+1}, x_j) such that $i+1 < j$ and $x_i < x_{i+1} < x_j$. Of course, $(ba)(w)$ denotes the number of *descents*, $\operatorname{des} w$ (i.e. the number of places $1 \le i < n$ such that $x_i > x_{i+1}$),

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and $(ab)(w)$ denotes the number of *rises*, $\mathrm{rise}\, w$ (i.e. the number of places $1 \le i < n$ such that $x_i < x_{i+1}$).

Using these atomic objects, Babson and Steingrímsson noticed that the classical permutation statistics "inv" and "maj" may be written as $(bc\text{-}a) + (ca\text{-}b) + (cb\text{-}a) + (ba)$ and $(a\text{-}cb) + (b\text{-}ca) + (c\text{-}ba) + (ba)$, respectively. This inspired them to perform a computer search for all statistics that could be thus written, and look for those that appear to be Mahonian. They came up with a list of 18. Some of them turned out to be well-known (e.g. "inv" and "maj"), and some were new, but they could do themselves. Yet eight new "conjecturally Mahonian" statistics were left open. Here we do four of them. The other four should also be amenable to our methods.

2. Notations and Results

Recall the usual notations

$$(2.1)\qquad (a;q)_n := \begin{cases} 1, & \text{if } n = 0; \\ (1-a)(1-aq)\,\ldots\,(1-aq^{n-1}), & \text{if } n \ge 1; \end{cases}$$

$$(2.2)\qquad (a;q)_\infty := \lim_n (a;q)_n = \prod_{n\ge 0} (1-aq^n).$$

Also, let

$$(2.3)\qquad [n]_q := \frac{1-q^n}{1-q} = (1+q+\cdots+q^{n-1});$$

$$(2.4)\qquad [n]_q! := \frac{(q;q)_n}{(1-q)^n} = [n]_q\,[n-1]_q\,\cdots\,[1]_q$$
$$= (1+q+\cdots+q^{n-1})(1+q+\cdots+q^{n-2})\cdots(1).$$

The sequence of polynomials $([n]_q!)$ $(n \ge 0)$ is said to be *Mahonian*. On the other hand, a statistic "stat" (actually, we should say a sequence (stat_n) $(n \ge 0)$ of statistics, where stat_n is defined on the symmetric group $\mathcal{S}_n$) is said to be *Mahonian*, if for every $n \ge 0$ we have

$$\sum_{w\in\mathcal{S}_n} q^{\mathrm{stat}\, w} = [n]_q!$$

A sequence $(A_n(t,q))$ $(n \ge 0)$ of polynomials in two variables t and q, is said to be *Euler-Mahonian*, if one of the following *equivalent* conditions holds:

(1) For every $n \ge 0$,

$$(2.5)\qquad \frac{1}{(t;q)_{n+1}}\, A_n(t,q) = \sum_{s\ge 0} t^s \left([s+1]_q\right)^n.$$

(2) The exponential generating function for the fractions $\dfrac{A_n(t,q)}{(t;q)_{n+1}}$ is given by:

$$\sum_{n\ge 0} \frac{u^n}{n!}\,\frac{A_n(t,q)}{(t;q)_{n+1}} = \sum_{s\ge 0} t^s\, \exp(u\,[s+1]_q). \tag{2.6}$$

(3) The sequence $(A_n(t,q))$ satisfies the recurrence relation:

$$(1-q)\,A_n(t,q) = (1-tq^n)\,A_{n-1}(t,q) - q(1-t)\,A_{n-1}(tq,q). \tag{2.7}$$

(4) Let $A_n(t,q) = \sum_{s\ge 0} t^s\, A_{n,s}(q)$. Then the coefficients $A_{n,s}(q)$ satisfy the recurrence:

$$A_{n,s}(q) = [s+1]_q\, A_{n-1,s}(q) + q^s[n-s]_q\, A_{n-1,s-1}(q). \tag{2.8}$$

It is routine to prove that those four conditions are equivalent (see, e.g. [ClFo95, §§ 6,7] for a proof in a more general setting). Now a pair of statistics $(\mathrm{stat}_1, \mathrm{stat}_2)$ defined on each symmetric group $\mathcal{S}_n$ $(n\ge 0)$ is said to be *Euler-Mahonian*, if for every $n\ge 0$ we have

$$\sum_{w\in\mathcal{S}_n} t^{\mathrm{stat}_1\, w} q^{\mathrm{stat}_2\, w} = A_n(t,q).$$

The first values of the Euler-Mahonian polynomials are the following: $A_0(t,q) = A_1(t,q) = 1$; $A_2(t,q) = 1+tq$; $A_3(t,q) = 1 + t(2q+2q^2) + t^2q^3$;
$A_4(t,q) = 1 + t(3q+5q^2+3q^3) + t^2(3q^3+5q^4+3q^5) + t^3q^6$;
$A_5(t,q) = 1 + t(4q+9q^2+9q^3+4q^4) + t^2(6q^3+16q^4+22q^5+16q^6+6q^7) + t^3(4q^6+9q^7+9q^8+4q^9) + t^4q^{10}$.

Our results are the following.

Theorem 1. *The permutation statistic*

$$\mathrm{S11} := (a\text{-}cb) + 2(b\text{-}ca) + (ba)$$

is Mahonian.

Theorem 2. *The permutation statistic*

$$\mathrm{S13} := (a\text{-}cb) + 2(b\text{-}ac) + (ab)$$

is Mahonian .

Theorem 3. *Let* $\mathrm{S5} := (b\text{-}ca) + (c\text{-}ab) + (a\text{-}bc) + (ab)$.
Then, the pair $(\mathrm{rise}, \mathrm{S5})$ *is Euler-Mahonian.*

Theorem 4. *Let* $\mathrm{S6} := (ba\text{-}c) + (c\text{-}ba) + (ba) + (ac\text{-}b)$.
Then, the pair $(\mathrm{des}, \mathrm{S6})$ *is Euler-Mahonian.*

Our Theorems 1, 2, and 4 are the three parts of Conjecture 8 of [BaSt00], while Theorem 3 is Conjecture 10 of [BaSt00]. In fact, our Theorem 4 is stronger, since in [BaSt00] S6 is only conjectured to be

Mahonian, but we will prove that it is in fact Euler-Mahonian when associated with "des".

Several pairs of statistics are known to be Euler-Mahonian, the pair $(\mathrm{des}, \mathrm{maj})$ being the Euler-Mahonian pair, *par excellence*, a result that goes back to Carlitz [Ca54], [Ca59], or even in an implicit form to MacMahon [Ma15, vol. 2, p. 211]. For proving that a given pair $(\mathrm{stat}_1, \mathrm{stat}_2)$ is Euler-Mahonian, we can form the generating polynomial for $\mathcal{S}_n$ by the pair $(\mathrm{stat}_1, \mathrm{stat}_2)$, say, $B_n(t,q)$ and show that one of the conditions (1), (2), (3) or (4) holds when $A_n(t,q)$ is replaced by $B_n(t,q)$. This is the approach we have adopted for proving the first three results, but we have done it in such a way that the calculation can be computer-implemented, as we now explain.

It turns out that, thanks to the second author's recent theory of the "Umbral Transfer Matrix Method" [Ze00], the proofs of the first three theorems are completely automatic, using the general Maple package `ROTA`, together with a new *interfacing* package `PERCY` that computes the appropriate Rota operators for what we will call *Markovian Permutation Statistics.*

Since *Markovian Permutation Statistics* are easily amenable to automatic treatment by `ROTA`, and S5, S11, and S13 happen to belong to that class (thanks to the consistent location of the 'dash'), all we did, after writing `PERCY`, was to 'plug them in'. `PERCY` can, actually, handle *arbitrary* Markovian Permutation Statistics. In particular, it can be used to prove, in a few nano-seconds, MacMahon's classical result that "maj" is Mahonian. We believe that it can also discover and prove many new results in addition to those proved here.

However, `ROTA` is useless in the case of S6, since the location of the dash in $\mathrm{S6} := (ac\text{-}b) + (ba\text{-}c) + (c\text{-}ba) + (ba)$ is not consistent. So proving Theorem 4 still requires the traditional combinatorial method: construct a *bijection* $w \mapsto w'$ of $\mathcal{S}_n$ onto itself which has the property that

$$(\mathrm{des}, \mathrm{S6})\, w' = (\mathrm{des}, \mathrm{maj})\, w$$

holds for every $w \in \mathcal{S}_n$. Instead of the pair $(\mathrm{des}, \mathrm{maj})$ we will take another Euler-Mahonian pair $(\mathrm{des}, \mathrm{mak})$, where "mak" is a Mahonian statistic that was introduced in our previous paper [FoZe90]. In the Babson-Steingrímsson notation "mak" reads

$$\mathrm{mak} := (a\text{-}cb) + (cb\text{-}a) + (ba) + (ca\text{-}b), \tag{2.9}$$

wrongly designated by "makl" in [BaSt00]. We will make use of the Euler-Mahonian pair $(\mathrm{des}, \mathrm{mak})$, but prove the stronger Theorem $4'$, where an involution of $\mathcal{S}_n$ is constructed that preverses the descent set and permutes two word statistics U and V we now define.

First, the *descent bottom set* of a permutation $w = x_1x_2 \ldots x_n$ is defined to be the set $\mathrm{DESBOT}\, w$ of all the x_i's such that $2 \le i \le n$ and $x_{i-1} > x_i$. Its cardinality is the *number of descents* of w, $\operatorname{des} w$, so that

$$\operatorname{des} w = \#\,\mathrm{DESBOT}\, w. \tag{2.11}$$

Next, the word statistics U and V are introduced as follows. Let $y = x_i$ be a letter of the permutation $w = x_1x_2 \ldots x_n$. Define

$$U_y(w) := (ca - b)\ |_{b\,=\,y}\ w; \tag{2.12}$$

$$V_y(w) := (b - ac)\ |_{b\,=\,y}\ w. \tag{2.13}$$

Thus, $U_y(w)$ is the number of pairs of adjacent letters x_jx_{j+1} to the left of $y = x_i$ such that $x_j > x_i > x_{j+1}$. The word statistics U and V are then:

$$U(w) := U_1(w)U_2(w)\ \ldots\ U_n(w); \tag{2.14}$$

$$V(w) := V_1(w)\, V_2(w)\ \ldots\ V_n(w). \tag{2.15}$$

Now, recall the traditional *reverse image* $\mathbf{r}$, which is an involution that maps each permutation $w = x_1x_2 \cdots x_n$ onto $\mathbf{r}\, w := x_n \cdots x_2x_1$. We shall introduce another involution $\mathbf{s}$ of $\mathcal{S}_n$, called the *rise-des-exchange*, which exchanges the rises and the descents of a permutation (in a sense that will be explained in Section 6, but can be immediately visualized in Fig. 1) and keeps peaks and troughs in their original ordering. We can then consider the involution $\mathbf{r}\,\mathbf{s}$.

Theorem 4′. *The involution* $\mathbf{r}\,\mathbf{s}$ *of* $\mathcal{S}_n$ *has the following properties:*
(a) $\mathrm{DESBOT}\,\mathbf{r}\,\mathbf{s}\, w = \mathrm{DESBOT}\, w$;
(b) $(U, V)\,\mathbf{r}\,\mathbf{s}\, w = (V, U)\, w$.

In Section 6 we will show how Theorem 4′ implies Theorem 4.

3. Markovian Permutation Statistics

The *reduction* of a sequence of n distinct integers w, denoted by $\operatorname{red}(w)$, is the permutation obtained by replacing the smallest member by 1, the second-smallest by 2, … , and the largest by n. For example, $\operatorname{red}(8\,3\,5\,9\,2) = 4\,2\,3\,5\,1$.

Definition. A permutation statistic $F : \mathcal{S}_n \to \mathbf{Z}$ is said to be *Markovian*, if there exists a function $h(j, i, n)$ such that

$$F(x_1 \cdots x_n) = F(\operatorname{red}(x_1 \cdots x_{n-1})) + h(x_{n-1}, x_n, n).$$

Definition: A Markovian permutation statistic $F : \mathcal{S}_n \to Z$ is said to be *Nice Markovian* if the above $h(j, i, n)$ can be written as

$$h(j, i, n) = \begin{cases} f(j, i, n), & \text{if } j < i; \\ g(j, i, n), & \text{if } j > i; \end{cases}$$

where f and g are *affine linear functions* of their arguments, i.e. can be written as $ai+bj+cn+d$, for some integers a,b,c,d.

¿From now on we, and the Maple package PERCY, will only consider Nice Markovian Permutation Statistics. We will denote them by $[f,g,j,i,n]$. For example, $\mathrm{inv}=[n-i,n-i,j,i,n]$, $\mathrm{maj}=[0,n-1,j,i,n]$, $\mathrm{des}=[0,1,j,i,n]$, $\mathrm{rise}=[1,0,j,i,n]$, etc.

Given a permutation statistic F we are interested in the sequence of polynomials

$$\mathrm{gf}(F)_n(q):=\sum_{w\in\mathcal{S}_n} q^{F(w)}\quad (n\geq 0).$$

However, in order to take advantage of Markovity, we need to consider the more refined

$$\mathrm{GF}(F)_n(q,z):=\sum_{w=x_1\ldots x_n\in\mathcal{S}_n} q^{F(w)}z^{x_n}\quad (n\geq 0)$$

that also keeps track of the last letter x_n. Now, using *Rota operators* (see [Ze00]), it is easy to express $\mathrm{GF}(F)_n$ in terms of $\mathrm{GF}(F)_{n-1}$. Let $w':=x'_1\cdots x'_{n-1}=\mathrm{red}(x_1\cdots x_{n-1})$; then

$$\begin{aligned}
\mathrm{GF}(F)_n(q,z) &:= \sum_{i=1}^{n} z^i \sum_{\substack{w\in\mathcal{S}_n\\ x_n=i}} q^{F(w)}\\
&= \sum_{i=1}^{n} z^i \Bigl(\sum_{j=1}^{i-1}\ \sum_{\substack{w\in\mathcal{S}_n\\ x_{n-1}=j,\, x_n=i}} q^{F(w)} + \sum_{j=i+1}^{n}\ \sum_{\substack{w\in\mathcal{S}_n\\ x_{n-1}=j,\, x_n=i}} q^{F(w)}\Bigr)\\
&= \sum_{i=1}^{n} z^i \Bigl(\sum_{j=1}^{i-1} \sum_{\substack{w\in\mathcal{S}_n\\ x_{n-1}=j}} q^{F(w')+f(j,i,n)} + \sum_{j=i+1}^{n} \sum_{\substack{w\in\mathcal{S}_n\\ x_{n-1}=j}} q^{F(w')+g(j,i,n)}\Bigr)\\
&= \sum_{j<i\leq n} z^i \sum_{\substack{w'\in S_{n-1}\\ x'_{n-1}=j}} q^{F(w')+f(j,i,n)} + \sum_{i\leq j\leq n-1} z^i \sum_{\substack{w'\in S_{n-1}\\ x'_{n-1}=j}} q^{F(w')+g(j+1,i,n)}\\
&= \sum_{j=1}^{n-1} \sum_{\substack{w'\in S_{n-1}\\ x'_{n-1}=j}} \Bigl(\sum_{i=1}^{j} q^{g(j+1,i,n)} z^i + \sum_{i=j+1}^{n} q^{f(j,i,n)} z^i\Bigr) q^{F(w')}.
\end{aligned}$$

Now for $1\leq j\leq n-1$ introduce the *umbra* $\mathcal{P}$,

$$\mathcal{P}(z^j):=\Bigl(\sum_{i=1}^{j} q^{g(j+1,i,n)} z^i + \sum_{i=j+1}^{n} q^{f(j,i,n)} z^i\Bigr), \tag{3.1}$$

and extend by linearity, so that $\mathcal{P}$ is defined on all polynomials of degree $\leq n-1$. In terms of $\mathcal{P}$, we have the very simple recurrence:

$$\mathrm{GF}(F)_n(q,z) = \mathcal{P}(\mathrm{GF}(F)_{n-1}(q,z)).$$

Of course, $\mathcal{P}$ itself may (and usually is) complicated, but *we don't care*, we don't even have to see it, since Maple can compute the umbra automatically, (all it has to do is to be able to sum geometric series symbolically, that it does very well), and store it for later. All the users have to enter is f and g, or if they prefer, they may use the notation of [BaSt00], and `PERCY` would convert it to the Markovian notation, and take it from there.

4. Proofs of Theorems 1, 2, 3

Proof of Theorem 1. Using `PERCY` (and `ROTA`) (or even by hand!) we get that the umbra $\mathcal{P}$ linking $\mathrm{GF}(\mathrm{S11})_{n-1}(q,z)$ to $\mathrm{GF}(\mathrm{S11})_n(q,z)$, as defined in (3.1), maps the polynomial $a(z)$ onto

$$\frac{z^{n+1}a(1) - za(z)}{z-1} + \frac{z(a(qz) - a(q^2))}{z-q}.$$

Hence $b_n(z) := \mathrm{GF}(\mathrm{S11})_n(q,z)$ satisfies the functional recurrence

$$b_n(z) = \frac{z^{n+1}b_{n-1}(1) - zb_{n-1}(z)}{z-1} + \frac{z(b_{n-1}(qz) - b_{n-1}(q^2))}{z-q},$$

with the initial condition $b_1(z) = z$. But, the same is true of

$$c_n(z) := z\,\frac{(z^n - q^n)}{(z-q)}[n-1]_q! \quad \text{(check!)}.$$

Hence $b_n(z) = c_n(z)$, and finally $b_n(1) = c_n(1) = [n]_q!$

Proof of Theorem 2. Using `PERCY` (and `ROTA`) (or even by hand!) we get that the umbra linking $\mathrm{GF}(\mathrm{S13})_{n-1}(q,z)$ to $\mathrm{GF}(\mathrm{S13})_n(q,z)$ is

$$a(z) \mapsto \frac{z(a(zq) - a(1))}{qz-1} + \frac{zqa(z) - q^{2n+1}z^{n+1}a(q^{-2})}{1-zq^2}.$$

Hence $d_n(z) := \mathrm{GF}(\mathrm{S13})_n(q,z)$ satisfies the functional recurrence

$$d_n(z) = \frac{z(d_{n-1}(zq) - d_{n-1}(1))}{qz-1} + \frac{zqd_{n-1}(z) - q^{2n+1}z^{n+1}d_{n-1}(q^{-2})}{1-zq^2}.$$

with the initial condition $d_1(z) = z$. But, the same is true of

$$e_n(z) := z\,\frac{(1 - q^n z^n)}{(1-qz)}[n-1]_q! \quad \text{(check!)}.$$

Hence $d_n(z) = e_n(z)$, and finally $d_n(1) = e_n(1) = [n]_q!$

Proof of Theorem 3. `PERCY` can just as easily compute the Umbra for *multi-statistics*, when the generating function is the weight-enumerator of $\mathcal{S}_n$ according to the weight

$$\mathrm{weight}(w) := z^{x_n} \prod_{j=1}^{r} q_j^{F_j(w)},$$

where $w = x_1 \ldots x_n$ and $F_1(w), \ldots, F_r(w)$ are *several* Nice Markovian permutation statistics. Define

$$A_n(t,q;z) := \sum_{w \in \mathcal{S}_n} t^{\mathrm{des}\, w} q^{\mathrm{maj}\, w} z^{x_n}, \quad B_n(t,q;z) := \sum_{w \in \mathcal{S}_n} t^{\mathrm{rise}\, w} q^{\mathrm{S5}\, w} z^{x_n}.$$

`PERCY`, using `ROTA`, computes the following functional equations (that in fact, are simple enough to be derivable by humans):

$$(4.1)\quad A_n(t,q;z) = \frac{z(1-tq^{n-1})A_{n-1}(t,q;z) - z(z^n - tq^{n-1})A_{n-1}(t,q;1)}{1-z};$$

$$(4.2)\quad B_n(t,q;z) = \frac{z(1-tq^{n})B_{n-1}(t,q;z) - z(1 - tz^{n})B_{n-1}(t,q;q)}{z-q}.$$

We have to prove that $B_n(t,q;1) = A_n(t,q;1) = A_n(t,q)$, the Euler-Mahonian polynomial, as defined in (2.5)-(2.8), as we already know that $(\mathrm{des}, \mathrm{maj})$ is Euler-Mahonian. By comparing the two functional recurrences, we get that $B_n(t,q;z) = q^{-n} z^{n+1} A_n(tq,q;q/z)$, hence we have to prove that $A_n(tq,q;q) = q^n A_n(t,q;1) = A_n(t,q)$. By plugging $t = tq$, $z = q$ into (4.1), and using the induction hypothesis, we get that

$$\begin{aligned}
A_n(tq,q;q)& \\
&= \frac{q(1-(tq)q^{n-1})A_{n-1}(tq,q;q) - q(q^n - (tq)q^{n-1})A_{n-1}(tq,q;1)}{1-q} \\
&= \frac{q(1-tq^{n})q^{n-1}A_{n-1}(t,q;1) - q^{n+1}(1-t)A_{n-1}(tq,q;1)}{1-q} \\
&= q^n \frac{(1-tq^{n})A_{n-1}(t,q) - q(1-t)A_{n-1}(tq,q)}{1-q}.
\end{aligned}$$

But, this equals $q^n A_n(t,q)$ by (2.7). □

The input and output files of `PERCY`, for all the above claimed statements, can be downloaded from

`http://www.math.temple.edu/~zeilberg/programs.html`.

5. A Brief Description of PERCY

You must have MAPLE^{TM}. First go to Zeilberger's homepage

`http://www.math.temple.edu/~zeilberg/`,

click on programs, and download PERCY and ROTA. Make sure that they reside in the same directory in your computer. Then go into Maple (by typing: `maple` , followed by ENTER), and type `read PERCY;` . Then follow the on-line instructions. In particular, to get a list of all procedures, type: `Ezra();` (not to be confused with `ezra();` and `ezra1();`, that refer to ROTA; for information about ROTA, see [Ze00]).

`BSE(bitui,j,i,n)` inputs an expression, `bitui` in the notation of [BaSt00], and three variables `j,i,n` and returns the Nice Markovian Statistic in the $[f, g, j, i, n]$ notation. Given a Markovian permutation statistic `MPS=[f,g,j,i,n]`, `PermPreUmbra(MPS,q,z);` and `PermUmbra(MPS,q,z);` find the pre-umbra and umbra respectively connecting the weight-enumerator of S_{n-1} to that of $\mathcal{S}_n$. For example `PermUmbra([0,n-1,j,i,n],q,z)` does it for *maj*.

Given a Markovian permutation statistic, MPS, a variable q, and a positive integer L, `GF(MPS,q,L);` finds the first L terms in the weight-enumeration sequence, *fast*, using the umbra (by calling ROTA). The skeptic can check the answer (for small L!), by also doing `GFdirect(MPS,q,L);`, which computes by direct (super-exponentially slow) enumeration. `PermUmbraMS(MPSs,qs,z)` and `GFMS(MPSs,qs,L)` are the analogous functions for multi-statistics, where MPSs is a list of Markovian permutation statistics and qs is a list of corresponding variables. For example, to get the first 10 terms in the sequence of the Euler-Mahonian polynomials, type `GFMS([[0,n-1,j,i,n],[0,1,j,i,n]],[q,t],10);` . For more details, download PERCY and use the on-line help.

6. Proof of Theorem 4′

The *reverse image* **r** was redefined at the end of Section 2. The definition of the involution **s**, which is now given, is based on the *peak-trough factorization* of a permutation. By *peak* of the permutation $w = x_1x_2\cdots x_n$ we mean a letter x_j such that $1 \le j \le n$ and $x_{j-1} < x_j$, $x_j > x_{j+1}$. [By convention, $x_0 = x_{n+1} := 0$.] By *trough* we mean a letter x_j such that $2 \le j \le n-1$ and $x_{j-1} > x_j$, $x_j < x_{j+1}$. Accordingly, each permutation has k troughs and $(k+1)$ peaks for some $k \ge 0$. By *double descent* we mean a letter x_j such that $2 \le j \le n$ and $x_{j-1} > x_j > x_{j+1}$. By *double rise* we mean a letter x_j such that $1 \le j \le n-1$ and $x_{j-1} < x_j < x_{j+1}$.

In the following p and t will designate *letters* which are peaks and troughs, respectively, while d, h, will designate *words* all letters of which are double descents, double rises, respectively.

Each permutation w has a unique factorization, called its *peak-trough factorization* $w = h_1p_1d_1t_1 \ \ldots \ h_{k-1}p_{k-1}d_{k-1}t_{k-1} \ h_kp_kd_k$, having the following properties:

(1) $k \geq 1$;

(2) the p_i's are the *peaks* and the t_i's the *troughs* of the permutation;

(3) for each i the symbols h_i, d_i are *words*, possibly empty, all letters of which are *double rises*, *double rises*, respectively.

It will be convenient to put $t_0 = t_k := 0$. The set of all the letters in the juxtaposition product $d_1t_1 \ldots d_{k-1}t_{k-1}d_k$, i.e. the set of all the double descents and troughs of the permutation w, is the *descent bottom set*, $\mathrm{DESBOT}\, w$, of w introduced in Section 2 (see (2.11)).

Definition of the involution $\mathbf{s}$. Let y be a *double rise* of w, so that y is a letter of a factor h_j for some j $(1 \leq j \leq k)$. Define $\varphi_w(y)$ to be the *least integer* i such that $j \leq i \leq k$ and $p_i > y > t_i$.

Define d'_i to be the *decreasing word* of all double rises y in w such that $\varphi_w(y) = i$. If $\varphi_w(y) \neq i$ for every double rise y of w, let d'_i be the empty word.

Let y be a double descent of w, so that y is a letter of a factor d_j for some j $(1 \leq j \leq k)$ and $p_j > y > t_j$. Define $\varphi_w(y)$ to be the *greatest integer* i such that $1 \leq i \leq j$ and $t_{i-1} < y < p_i$.

Define h'_i to be the *increasing word* of all double descents y in w such that $\varphi_w(y) = i$. If $\varphi_w(y) \neq i$ for every double descent y of w, let h'_i be the empty word.

Then, let

$$\mathbf{s}\, w := h'_1p_1d'_1t_1 \ \ldots \ h'_{k-1}p_{k-1}d'_{k-1}t_{k-1} \ h'_kp_kd'_k. \tag{6.1}$$

The construction is illustrated in Fig. 1. To obtain the graph of $\mathbf{s}\, w$ we have to move the double rises to the right until they hit the first decreasing segment $p_i > t_i$. There is necessarily one since, by assumption, $t_k = 0$. Likewise, we move the double descents to the left until they reach the first increasing segment $t_{i-1} < p_i$, which exists since $t_0 = 0$ by convention.

In the example we start with the permutation

$$w = 10, \overset{\wedge}{11}, \underset{\vee}{2}, 4, 5, \overset{\wedge}{9}, \underset{\vee}{6}, 12, 14, \overset{\wedge}{15}, 7, 3, \underset{\vee}{1}, 8, \overset{\wedge}{13}$$

and we obtain the permutation

$$\mathbf{s}\, w = \overset{\wedge}{11}, 10, \underset{\vee}{2}, 3, \overset{\wedge}{9}, \underset{\vee}{6}, 7, \overset{\wedge}{15}, 14, 12, 5, 4, \underset{\vee}{1}, \overset{\wedge}{13}, 8,$$

where the peaks (resp. the troughs) are materialized by "$\wedge$" (resp. "$\vee$")

Let $\mathrm{RISE}\, w$ (resp. $\mathrm{DES}\, w$, resp. $\mathrm{TROUGH}\, w$) be the set of of all the double rises (resp. the double descents, resp. the troughs) of w, so that $\mathrm{DESBOT}\, w = \mathrm{DES}\, w \cup \mathrm{TROUGH}\, w$.

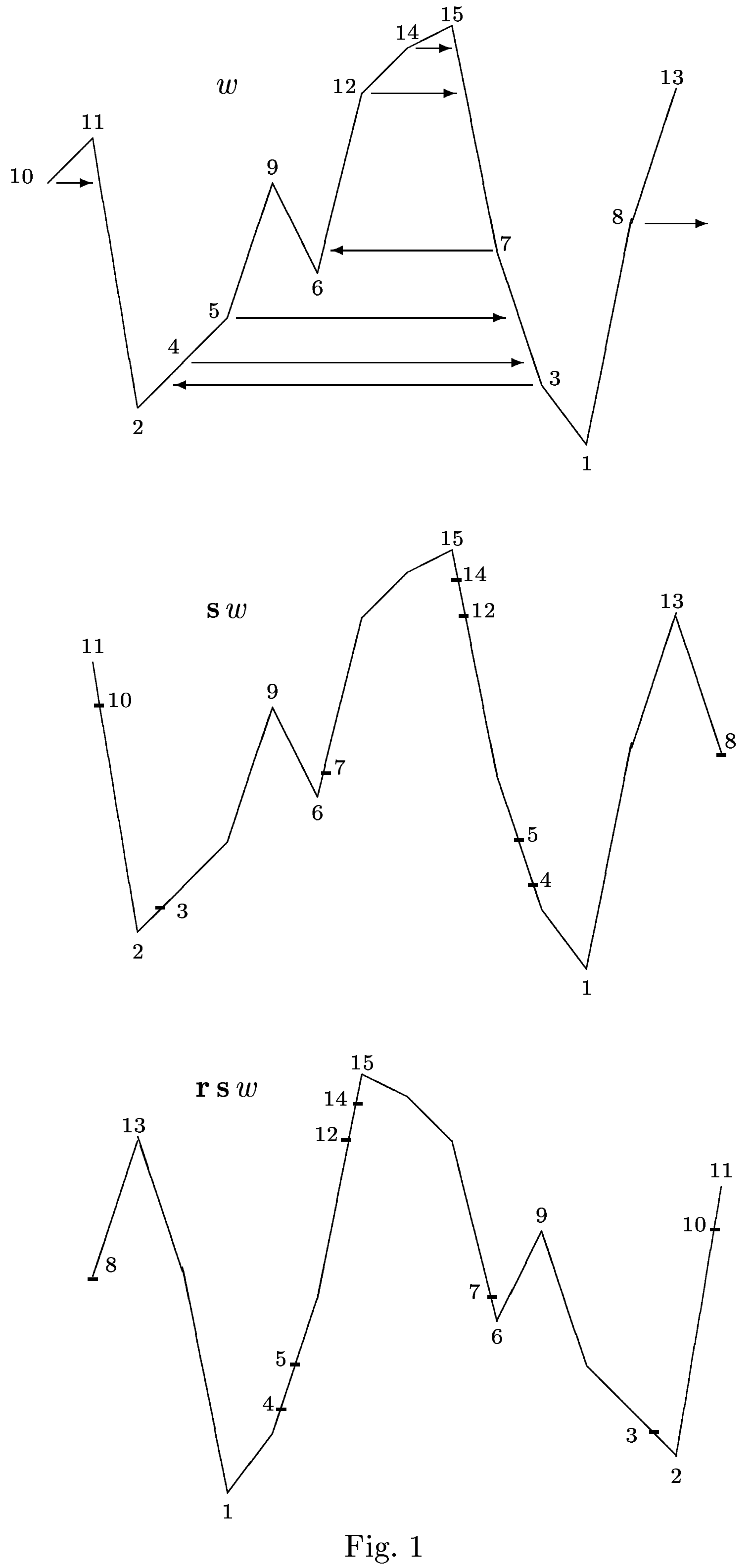

Fig. 1

Lemma 6.1. *The operation* $\mathbf{s}$ *is an involution. Moreover,*
(a) $\text{DES}\,\mathbf{s}\,w = \text{RISE}\,w$, $\text{RISE}\,\mathbf{s}\,w = \text{DES}\,w$ *and* $\text{TROUGH}\,\mathbf{s}\,w = \text{TROUGH}\,w$;
(b) $(U,V)\,\mathbf{s}\,w = (U,V)\,w$.

Proof. When passing from w to $\mathbf{s}\,w$ the subword $p_1t_1\ \ldots\ p_{k-1}t_{k-1}\,p_k$ made of the successive peaks and troughs remains alike. The double rises (resp. double descents) of w are transformed into double descents (resp. double rises) of $\mathbf{s}\,w$.

By considering the graphs of the permutations w and $\mathbf{s}\,w$ it is clear that we recover w by applying $\mathbf{s}$ to $\mathbf{s}\,w$ itself.

Finally, if y is a double rise of w between t_{j-1} and p_j, we have defined $\varphi_w(y)$ as the *least integer* i such that $j \le i \le k$ and $p_i > y > t_i$. Accordingly, the number of pairs $p_l > t_l$ to the left of y in both w and $\mathbf{s}\,w$ is the same and the number of pairs $t_{l-1} < p_l$ to the right of y is also the same in both w and $\mathbf{s}\,w$. Accordingly,

$$\begin{aligned}
U_y(w) &:= (ca\text{-}b)\,|_{b\,=\,y}\; w\\
&= \sum_{1\le l\le j-1} \chi(p_l > y > t_l)\\
&= \sum_{1\le l\le i-1} \chi(p_l > y > t_l)\\
&= (ca\text{-}b)\,|_{b\,=\,y}\; \mathbf{s}\,w = U_y(\mathbf{s}\,w).
\end{aligned}$$

In the same manner,

$$\begin{aligned}
V_y(w) &:= (b\text{-}ac)\,|_{b\,=\,y}\; w\\
&= \sum_{j+1\le l\le k-1} \chi(t_l < y < p_{l+1})\\
&= \sum_{i\le l\le k-1} \chi(t_l < y < p_{l+1})\\
&= (b\text{-}ac)\,|_{b\,=\,y}\; \mathbf{s}\,w = V_y(\mathbf{s}\,w).
\end{aligned}$$

There is an analogous proof when y is a double descent of w. □

Lemma 6.2. *The reverse image* $\mathbf{r}$ *has the following properties:*
(a) $\text{DES}\,\mathbf{r}\,w = \text{RISE}\,w$, $\text{RISE}\,\mathbf{r}\,w = \text{DES}\,w$ *and* $\text{TROUGH}\,\mathbf{r}\,w = \text{TROUGH}\,w$;
(b) $(U,V)\,\mathbf{r}\,w = (V,U)\,w$.

Proof. Property (a) is obvious. On the other hand, when going from w to $\mathbf{r}\,w$ each decrease ca to the left of y in w will become an increase ac to the right of y in $\mathbf{r}\,w$, so that (b) holds. □

It follows from Lemmas 6.1 and 6.2 that $\text{DES}\,\mathbf{r}\,\mathbf{s}\,w = \text{RISE}\,\mathbf{s}\,w = \text{DES}\,w$ and $\text{TROUGH}\,\mathbf{r}\,\mathbf{s}\,w = \text{TROUGH}\,\mathbf{s}\,w = \text{TROUGH}\,w$, so that $\text{DESBOT}\,\mathbf{r}\,\mathbf{s}\,w = \text{DESBOT}\,w$. Finally, $(U,V)\,\mathbf{s}\,\mathbf{r}\,w = (U,V)\,\mathbf{r}\,w = (V,U)\,w$. This completes the proof of Theorem 4′.

In the example of Fig. 1 we have

$$\begin{aligned}
w &= 10, \overset{\wedge}{11}, \underset{\vee}{2}, 4, 5, \overset{\wedge}{9}, \underset{\vee}{6}, 12, 14, \overset{\wedge}{15}, 7, 3, \underset{\vee}{1}, 8, \overset{\wedge}{13},\\
\mathbf{s}\, w &= \overset{\wedge}{11}, 10, \underset{\vee}{2}, 3, \overset{\wedge}{9}, \underset{\vee}{6}, 7, \overset{\wedge}{15}, 14, 12, 5, 4, \underset{\vee}{1}, \overset{\wedge}{13}, 8,\\
\mathbf{r}\,\mathbf{s}\, w &= 8, \overset{\wedge}{13}, \underset{\vee}{1}, 4, 5, 12, 14, \overset{\wedge}{15}, 7, \underset{\vee}{6}, \overset{\wedge}{9}, 3, \underset{\vee}{2}, 10, \overset{\wedge}{11},
\end{aligned}$$

and we can verify that:

$$\begin{pmatrix} y \\ U_y(\mathbf{r}\,\mathbf{s}\, w) \\ V_y(\mathbf{r}\,\mathbf{s}\, w) \end{pmatrix} = \begin{pmatrix} 1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ 0&1&1&1&1&1&1&0&2&2&2&1&0&0&0\\ 0&0&1&1&1&1&2&3&1&0&0&0&1&0&0 \end{pmatrix} = \begin{pmatrix} y \\ V_y(w) \\ U_y(w) \end{pmatrix}.$$

7. Proof of Theorem 4 and Consequences

Let $\Sigma\,\mathrm{DESBOT}\, w$ be the *sum* of all the letters x_i of the permutation $w = x_1 x_2 \cdots x_n$ which belong to the descent bottom set $\mathrm{DESBOT}\, w$.

Lemma 7.1. *For each permutation w we have:*

$$\Sigma\,\mathrm{DESBOT}\, w = \big((a\text{-}cb) + (cb\text{-}a) + (ba)\big)\, w. \tag{7.1}$$

Proof. Let $x_i \in \mathrm{DESBOT}\, w$, so that $x_{i-1} > x_i$ and $(ba)\,|_{b\,=\,x_i}\, w = 1$. Then $(a\text{-}cb)\,|_{b\,=\,x_i}\, w$ counts the letters x_j, less than x_i, which are to the left of x_i, while $(cb\text{-}a)\,|_{b\,=\,x_i}\, w$ counts the letters x_j, less than x_i, to the right of x_i. Hence, $\big((a\text{-}cb) + (cb\text{-}a) + (ba)\big)\,|_{b\,=\,x_i}\, w$ is equal to x_i. □

Next, introduce the other classical operation of the dihedral group, the *complement to* $(n+1)$, denoted by $\mathbf{c}$, that maps each permutation $w = x_1 x_2 \ldots x_n$ onto $\mathbf{c}\, w := (n+1-x_1)(n+1-x_2)\cdots(n+1-x_n)$. It is straightforward to verify the relations: $(ba\text{-}c)\,\mathbf{r}\,\mathbf{c} = (a\text{-}cb)$, $(c\text{-}ba)\,\mathbf{r}\,\mathbf{c} = (cb\text{-}a)$, $(ac\text{-}b)\,\mathbf{r}\,\mathbf{c} = (b\text{-}ac)$, $(ba)\,\mathbf{r}\,\mathbf{c} = (ba)$.

As stated in Theorem 4, the statistic S6 is defined by

$$\mathrm{S6} := (ba\text{-}c) + (c\text{-}ba) + (ba) + (ac\text{-}b),$$

so that the statistic $\mathrm{S6}\,\mathbf{r}\,\mathbf{c}$ reads

$$\mathrm{S6}\,\mathbf{r}\,\mathbf{c} := (a\text{-}cb) + (cb\text{-}a) + (ba) + (b\text{-}ac),$$

while the "mak" statistic defined in (2.9) is

$$\mathrm{mak} = (a\text{-}cb) + (cb\text{-}a) + (ba) + (ca\text{-}b).$$

Taking Lemma 7.1 into account, as well as the definitions of the word statistics U, V given in (2.12)–(2.15), we get the expressions:

$$\begin{aligned}
\mathrm{mak}, w &= \Sigma\,\mathrm{DESBOT}\, w + U_1(w) + \cdots + U_n(w);\\
\mathrm{S6}\,\mathbf{r}\,\mathbf{c}\, w &= \Sigma\,\mathrm{DESBOT}\, w + V_1(w) + \cdots + V_n(w).
\end{aligned}$$

Now, remember that $\mathrm{des}\, w$ is the number of elements in $\mathrm{DESBOT}\, w$. Therefore, Theorem 4′ implies the following corollary.

Corollary 7.2. *The involution* $\mathbf{r}\,\mathbf{s}$ *is an involution of* $\mathcal{S}_n$ *having the property:*

$$(\mathrm{des}, \mathrm{mak})\, w = (\mathrm{des}, \mathrm{S6}\,\mathbf{r}\,\mathbf{c})\, rs\, w.$$

But $(\mathrm{des}, \mathrm{mak})$ *is Euler-Mahonian, as proved in [FoZe90]. Therefore, the pair* $(\mathrm{des}, \mathrm{S6}\,\mathbf{r}\,\mathbf{c})$ *is Euler-Mahonian, as well as* $(\mathrm{des}, \mathrm{S6})$*, since we always have* $\mathrm{des}\,\mathbf{r}\,\mathbf{c}\, w = \mathrm{des}\, w$*. Hence Theorem 4 is proved.*

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