How Likely Is Pólya's Drunkard to Return to the Pub Without Getting Mugged? (In d-Dimensional Manhattan $[d \geq 2]$)

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In 1921, George Pólya[P] (see also [DS], ch. 7) famously proved that if a drunkard leaves a pub situated at the origin, in a d-dimensional Manhattan (w/o Broadway), and where he is allowed to only walk on streets and avenues, then he will return to the pub with probability 1 if $d \leq 2$, but with probability < 1 if $d > 2$. Later writers (see [F], sec. 5.9) computed (calling this probability p_d), that $p_3 = .340537\ldots$, $p_4 = .193201\ldots$, $p_5 = .135178\ldots$, $p_6 = .104715\ldots$, etc.

But Pólya unrealistically assumed that it is safe to walk everywhere in Manhattan. As we all know, in some areas, a person-especially if he looks drunk-is *guaranteed* to get mugged.

In 1989, Jet Wimp and I ([WimZ]) determined that, in a 3-dimensional Manhattan, where it is only safe to walk in $\{(x_1, x_2, x_3) \in Z^3 \,|\, x_1 \geq x_2 \geq x_3\}$, the probability of returning to the pub, *without getting mugged*, is $.0648447\ldots$.

But what about other dimensions and other regions? In this note, we will answer the following questions.

What are the probabilities of a simple random walker, starting at the origin of Z^d, and walking with unit positive steps, to return to the origin and stay in:

(a) $\{(x_1, \ldots, x_d) \in Z^d \,|\, x_1 \geq 0\,,\, x_2 \geq 0\,,\, \ldots\,,\, x_d \geq 0\,\}$,

(b) $\{(x_1, \ldots, x_d) \in Z^d \,|\, x_1 \geq \ldots \geq x_d\,\}$,

(c) $\{(x_1, \ldots, x_d) \in Z^d \,|\, x_1 \geq \ldots \geq x_d \geq 0\,\}$?

Calling these quantities a_d, b_d, c_d, we found, using the Maple package `DRUNKARD` accompanying this article, that

$a_2 = 0.1731362517\ldots$, $b_2 = 0.2146018366\ldots$, $c_2 = 0.07422872309\ldots$,

$a_3 = 0.1019012545\ldots$, $b_3 = 0.064844715377\ldots$, $c_3 = 0.0295951553088\ldots$,

$a_4 = 0.072168388084\ldots$, $b_4 = 0.0336123617\ldots$, $c_4 = 0.01615664624\ldots$,

$a_5 = 0.05590880956867\ldots$, $b_5 = 0.020894826887\ldots$, $c_5 = 0.01021082007\ldots$.

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Dec. 17, 2007. Accompanied by the Maple package `DRUNKARD` downloadable from Zeilberger's website. Exclusively published in the Personal Journal of Ekhad and Zeilberger `http://www.math.rutgers.edu/~zeilberg/pj.html`. Supported in part by the NSF.

(Note that b_3 was the subject matter of [WimZ].)

DRUNKARD

The Maple package `DRUNKARD` "accompanying" this article (to be fair, this article accompanies the package) has lots of interesting related features.

One thing it can also do is compute the probability of making it *home* from the pub without getting mugged. See procedures `Pr0, Pr1, Pr2, Pr3`. `Pr0` handles the unrestricted case, while `Pr1, Pr2, Pr3` take care of the above-mentioned regions $(a), (b), (c)$ respectively. For example, "`Pr3([3,2],[6,4]);`" computes the probability of leaving point $[3,2]$ and eventually making it to point $[6,4]$ without ever going outside of $\{(x,y)|x \geq y \geq 0\}$. By the way, it happens to be $\%6.887\ldots$.

`DRUNKARD` also computes the sequences enumerating the number of n-step walks starting at S and ending at F (for any S and F, in particular where both are the origin) both unrestricted, and staying in each of the above regions. In fact, it computes them in *three* different ways. The first way is by implementing the obvious recurrences, the second way is by using the Gessel-Zeilberger[GeZ] Constant Term Expressions (equivalent to trigonometric integrals), while the third way is via exponential generating functions, using Grabiner-Magyar-style formulas (the package computes them *ab initio*, without actually entering their formulas. All we used were the *ideas*, that we taught to our computer). They all agree! Which means that the probability that they are all correct is very high.

Mathematics

Strictly speaking, there is no *new* mathematics in this note, its novelty is in the (hopefully) **efficient** implementation of already existing mathematics, that can be found in [GeZ], [GrM], and in the references of [F], sec. 5.9.

The readers are very welcome to browse the *source code* of `DRUNKARD`, read the documentation, and experiment with it.

Let me just briefly describe how `DRUNKARD` computes the above probabilities of return. Just like in the unrestricted case, if $f(t)$ is the exponential generating function for the sequence enumerating n-step walks, from and to the origin, obeying the restrictions, then the *expected number of visits*, m, is $\int_0^\infty f(t/(2k))e^{-t}\,dt$ (why?). If the probability of return is p, then of course $m = 1/(1-p)$ and so $p = 1 - 1/m$. The exponential generating function, in each case, is expressible as a certain polynomial expression in various Modified Bessel functions, $J_m(t) := \sum_{k=0}^{\infty} \frac{t^k}{k!(m+k)!}$ (see [GrM]), and that expression is computed automatically by our Maple package.

Sample Input and Output

The webpage of this article

http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/drunkard.html

has some input and output. The readers can generate much more on their own.

Asymptotics

If $M_0(d;n), M_1(d;n), M_2(d;n), M_3(d;n)$ are the number of ways of walking from the origin, back to the origin, in $2n$ steps, in Z^d, $\{(x_1, \dots, x_d) \in Z^d | x_1 \geq 0, \dots, x_d \geq 0\}$, $\{(x_1, \dots, x_d) \in Z^d | x_1 \geq x_2 \geq \dots \geq x_d\}$, and $\{(x_1, \dots, x_d) \in Z_d | x_1 \geq x_2 \geq \dots \geq x_d \geq 0\}$, respectively, for fixed d, and as $n \to \infty$, we have the asymptotics

$$\frac{M_0(d;n)}{(2d)^{2n}} \sim C_0(d) \cdot n^{-d/2} \quad , \quad \frac{M_1(d;n)}{(2d)^{2n}} \sim C_1(d) \cdot n^{-3d/2} \quad ,$$

$$\frac{M_2(d;n)}{(2d)^{2n}} \sim C_2(d) \cdot n^{-d^2/2} \quad , \quad \frac{M_3(d;n)}{(2d)^{2n}} \sim C_3(d) \cdot n^{-(d^2+d/2)} \quad .$$

These can be obtained, using standard asymptotic methods, from the integral representation formulas alluded to at the end of [GeZ]. We leave the details to the interested readers.

References

[DS] P.G. Doyle and J.L. Snell, *"Random Walks and Electric Networks"*, Math. Assoc. of America, 1984.

[F] S.R. Finch, *"Mathematical Constants"*, Cambridge University Press, 2003.

[GeZ] I. Gessel and D. Zeilberger, *Random Walk in a Weyl chamber*, Proc. Amer. Math. Soc. **115**, 27-31 (1992).

[GrM] D. Grabiner and P. Magyar, *Random Walks in a Weyl Chamber and the decomposition of tensor powers*, J. Alg. Combinatorics **2** (1993), 239-260.

[P] G. Pólya, *Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Stassennetz*, Math. Annalen **84** (1921), 149-160. Also in *"Collected Papers"*, v. 4, ed. G. -C. Rota, MIT Press, 1984, 69-80, 609.

[WimZ] J. Wimp and D. Zeilberger, *How likely is Polya's drunkard to stay in $x \geq y \geq z$? J. Stat. Physics* **57** *(1989), 1129-1135.*