Using GENERATINGFUNCTIONOLOGY to Enumerate Distinct-Multiplicity Partitions

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In fond memory of Guru Herbert Saul WILF (28 Sivan 5691- 12 Tevet 5772) zecher gaon l'bracha

Preamble

About a year ago, Herb Wilf[W1] posed, on-line, eight intriguing problems. I don't know the *answer* to any of them, but I will say something about the sixth question.

Herb Wilf's 6th Question: *Let $T(n)$ be the set of partitions of n for which the (nonzero) multiplicities of its parts are all different, and write $f(n) = |T(n)|$. See Sloane's sequence* **A098859** *for a table of values. Find any interesting theorems about $f(n)$ ...*

First, I will explain how to compute the first few terms of $f(n)$. Shalosh can easily get the first 250 terms, but as n gets larger it gets harder and harder to compute, unlike its unrestricted cousin, $p(n)$. I conjecture that the fastest algorithm takes exponential time, but I have no idea how to prove that claim. I am impressed that, according to Sloane, Maciej Ireneusz Wilczynsk computed 508 terms.

Recall that the generating function for the number of integer partitions of n whose largest part is $\leq m$, $p_m(n)$, is the very simple rational function

$$\sum_{n=0}^{\infty} p_m(n) q^n = \frac{1}{(1-q)(1-q^2)\cdots(1-q^m)} \quad .$$

The main purpose of this note is to describe, using *Generatingfunctionology*, so vividly and lucidly *preached* in W's **classic book** [W2], how to compute the generating function (that also turns out to be rational) for the number of partitions of n whose largest part is $\leq m$ **and** all its (nonzero) multiplicities are distinct, let's call it $f_m(n)$. As m gets larger, the formulas get more and more complicated, but we sure do have an **answer**, in the sense of the **classic article** [W3], for any *fixed* m, but of course not for a *symbolic* m.

Even more is true! Because, like $\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}$, the generating function of $f_m(n)$, $\sum_{n=0}^{\infty} f_m(n) q^n$, turns out (as we will see) to only have roots-of-unity poles, whose highest order is m, it follows

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`http://www.math.rutgers.edu/~zeilberg/` . First Version: Jan. 18, 2012. Last version, Feb. 4, 2012 [putting a link to Vaclav Kotesovec's graph. Pervious Version: Feb. 1, 2012 [adding a comment of Daniel Kane at the end]. Accompanied by Maple package `DMP`
downloadable from `http://www.math.rutgers.edu/~zeilberg/tokhniot/DMP` . Sample input and output files may be viewed in the front of this article:
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/dmp.html` . Supported in part by the USA National Science Foundation.

that $f_m(n)$ is a *quasi-polynomial* of degree $m-1$ in n. Now that's a very good answer! (in W's sense, albeit only for a fixed m).

How to Compute Many terms of $f(n)$?

$p_m(n)$ is very easy to compute. For example, one may use the recurrence

$$p_m(n) = p_{m-1}(n) + \sum_{i=1}^{\lfloor n/m \rfloor} p_{m-1}(n-mi) \quad ,$$

together with the *initial condition* $p_1(n) = 1$, $p_m(0) = 1$.

How can we adapt this in order to compute $f_m(n)$? The contribution from the partitions counted by $f_m(n)$ where m does not show up is $f_{m-1}(n)$, in analogy with the $p_{m-1}(n)$ term in the above recurrence. But if m *does* show up, it does so with a certain multiplicity, i, say, where $1 \le i \le \lfloor n/m \rfloor$, and removing these i copies of m results in a partition counted by $f_{m-1}(n-mi)$ -so all its multiplicities are different- and **in addition** none of these multiplicities may be i. Continuing, we are forced to introduce a much more general discrete function $f_m(n;S)$ whose arguments are m and n and a set of "forbidden multiplicities", S.

So let's define $f_m(n;S)$ to be the number of partitions of n with parts $\le m$, with all its multiplicities distinct **and** none of these multiplicities belonging to S. Our intermediate object of desire, $f_m(n)$, is simply $f_m(n;\emptyset)$, and the ultimate object, $f(n)$, is $f_n(n;\emptyset)$.

The recurrence for $f_m(n;S)$ is, naturally,

$$f_m(n;S) = f_{m-1}(n;S) + \sum_{i=1, i \notin S}^{\lfloor n/m \rfloor} f_{m-1}(n-im; S \cup \{i\}) \quad ,$$

because once we decided on the number of times m shows up, let's call it i, where i is between 1 and $\lfloor m/m \rfloor$ and $i \notin S$, the partition (of $n-mi$) obtained by removing these i copies of m must forbid the set of multiplicities $S \cup \{i\}$.

In the Maple package `DMP`, procedure `qnmS(n,m,S)` implements $f_m(n;S)$ and procedure `qn(n)` implements $f(n)$.

Inclusion-Exclusion

Let $\mathcal{P}_m(n)$ be the set of partitions of n whose parts are all $\le m$, in other words, the set that $p_m(n)$ is counting. Consider the set of *all* partitions whose largest part is $\le m$, where we write a partition in *frequency notation*:

$$\mathcal{P}_m := \{1^{a_1} 2^{a_2} \dots m^{a_m} \,|\, a_1, \dots, a_m \ge 0\} \quad .$$

For example $1^3 2^5 4^2$ is the partition of twenty-one usually written as 4422222111. Introducing *symbols* $x_1, x_2, \dots, x_m$, we define the *Weight* of a partition to be

$$Weight(1^{a_1} 2^{a_2} \dots m^{a_m}) := x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m} \quad .$$

The *weight-enumerator* of $\mathcal{P}_m$ is, by **ordinary**-generatingfunctionology

$$Weight(\mathcal{P}_m) = \frac{1}{(1-x_1)(1-x_2)\cdots(1-x_m)} \quad ,$$

since we make m *independent* decisions:

• how many copies of 1?(Weight enumerator $= 1 + x_1 + x_1^2 + \ldots = (1-x_1)^{-1}$) ,

• how many copies of 2? (Weight enumerator $= 1 + x_2 + x_2^2 + \ldots = (1-x_2)^{-1}$) ,

...

• how many copies of m? (Weight enumerator $= 1 + x_m + x_m^2 + \ldots = (1-x_m)^{-1}$).

But we want to find the weight-enumerator of the much-harder-to-weight-count set

$$\mathcal{F}_m := \{1^{a_1}2^{a_2}\ldots m^{a_m} \,|\, a_1, \ldots, a_m \geq 0\,;\, a_i \neq a_j \ \ (if \quad a_i > 0, a_j > 0)\} \quad .$$

Calling the members of $\mathcal{F}_m$ *good*, we see that a member of $\mathcal{P}_m$ is good if it does **not** belong to any of the following $\binom{m}{2}$ sets, S_{ij} $1 \leq i < j \leq m$:

$$S_{ij} := \{1^{a_1}2^{a_2}\ldots m^{a_m} \in \mathcal{P}_m \,|a_i = a_j > 0\} \quad .$$

By inclusion-exclusion, the weight-enumerator of $\mathcal{F}_m$ is

$$\sum_G (-1)^{|G|} Weight\left(\mathop{\cap}_{ij \in G} S_{ij} \right) \quad ,$$

where the summation ranges over **all** $2^{m(m-1)/2}$ subsets of $\{(i,j) \,|\, 1 \leq i < j \leq m\}$.

But the G's can be naturally viewed as *labeled graphs* on m vertices. Such a graph has several connected components, and together they naturally induce a *set partition* $\{C_1, C_2, \ldots, C_r\}$ of $\{1, 2, \ldots, m\}$. We have:

$$Weight\left(\mathop{\cap}_{ij \in G} S_{ij} \right) = \prod_{i=1}^{r} weight(C_i) \quad ,$$

where if $|S| = 1$, $S = \{s\}$, say, then $weight(S) = \frac{1}{1-x_s}$, and if $|S| = d > 1$, $S = \{s_1, s_2, \ldots s_d\}$, say, then

$$weight(S) = \frac{x_{s_1}x_{s_2}\cdots x_{s_d}}{1 - x_{s_1}x_{s_2}\cdots x_{s_d}} \quad .$$

To justify the latter, note that if vertices $s_1, s_2, \ldots s_d$ all belong to the same connected component of our graph then, by *transitivity*, we have that all $a_{s_1} = a_{s_2} = \ldots = \ldots a_{s_d} > 0$, and the weight-enumerator is the infinite geometric series

$$\sum_{\alpha=1}^{\infty} (x_{s_1}\cdots x_{s_d})^{\alpha} = \frac{x_{s_1}x_{s_2}\cdots x_{s_d}}{1 - x_{s_1}x_{s_2}\cdots x_{s_d}} \quad .$$

But quite a few graphs correspond to any one set-partition. To find out the coefficients in front, for any set-partition $\{C_1, C_2, \dots, C_r\}$ of $\{1, \dots, m\}$ we must find

$$\sum_G (-1)^{|G|} \quad ,$$

summed over all the graphs that gives rise to the above set partition. But this is the product of the analogous sums where one focuses on one connected component at a time, and then multiplies everything together.

Let's digress and figure out $\sum_G (-1)^{|G|}$ over all *connected labeled graphs* on n vertices. For the sake of clarity, let's, more generally, figure out $\sum_G y^{|G|}$ with a general variable y.

By **exponential**-generatingfunctionology[W2] (see also [Z]), this sum is nothing but the coefficient of $t^n/n!$ in

$$\log\left(\sum_{i=0}^{\infty} (1+y)^{\binom{i}{2}} \frac{t^i}{i!}\right) \quad .$$

Going back to $y = -1$, we see that we need the coefficient of $t^n/n!$ in

$$\log\left(\sum_{i=0}^{\infty} (1+(-1))^{\binom{i}{2}} \frac{t^i}{i!}\right) = \log\left(\sum_{i=0}^{\infty} 0^{\binom{i}{2}} \frac{t^i}{i!}\right) = \log(1+t)$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} (n-1)! \frac{t^n}{n!} \quad .$$

So the desired sum is $(-1)^{n-1}(n-1)!$.

Let's define for any set of positive integers, S,

$$mishkal(S) = \begin{cases} 1/(1-x_s), & \text{if } |S| = 1 \text{ where } S = \{s\} \ ; \\ (-1)^{d-1}(d-1)!(x_{s_1} \cdots x_{s_d})/(1 - x_{s_1} \cdots x_{s_d}), & \text{if } |S| = d > 1 \text{ where } S = \{s_1, \dots s_d\}. \end{cases}$$

For any set partition $C = \{C_1, \dots, C_r\}$ let's define

$$Mishkal(C) = mishkal(C_1) \cdots mishkal(C_r) \quad .$$

It follows that the weight-enumerator of $\mathcal{F}_m$ according to $Weight(1^{a_1} 2^{a_2} \dots m^{a_m}) := x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}$ is

$$\sum_C Mishkal(C) \quad ,$$

where the sum has B_m terms (B_m being the Bell numbers), one for each set-partition of $\{1, \dots, m\}$.

Finally, to get an "explicit" formula (as a sum of B_m terms, each a simple rational function of q), for the generating function $\sum_{n=0}^{\infty} f_m(n) q^n$, all we need is replace x_i by q^i, for $i = 1 \dots m$, getting

$$\sum_{n=0}^{\infty} f_m(n) q^n = \sum_C Poids(C) \quad ,$$

where for a set partition $C = \{C_1, \ldots, C_r\}$

$$Poids(C) = poids(C_1) \cdots poids(C_r) \quad ,$$

and where for an individual set S:

$$poids(S) = \begin{cases} 1/(1-q^s), & \text{if } |S| = 1 \text{ where } S = \{s\} \text{ ;} \\ (-1)^{d-1}(d-1)! q^{s_1+\ldots+s_d}/(1-q^{s_1+\ldots+s_d}), & \text{if } |S| = d > 1 \text{ where } S = \{s_1, \ldots s_d\}. \end{cases} \quad .$$

It follows that indeed $f_m(n)$ is a quasi-polynomial of degree $m-1$ in n. Furthermore, since the only pole that has multiplicity m is $q = 1$, it follows that the leading term (of degree $m-1$) is a pure polynomial.

The generating function, $\sum_{n=0}^{\infty} f_m(n) q^n$, for any desired positive integer m, is implemented in procedure `GFmq(m,q)` in the Maple package `DMP`. For the Weight-enumerator (or rather with x_i replaced by $q^i x_i$, for $i = 1, \ldots, m$), see `GFmxq(m,x,q)`. Since the Bell numbers grow very fast, the formulas get complicated rather fast, but *in principle* we do have a very nice *answer* for any specific m, but *in practice*, for large m it is only "nice" in principle. Of course it is *anything but* nice when viewed also as function of m, and that's why $f(n) = f_n(n)$ is **probably** very hard to compute for larger n.

To see the outputs of `GFmq(m,q)` for $1 \leq$ `m` ≤ 8 see:
`http://www.math.rutgers.edu/~zeilberg/tokhniot/oDMP3` .

Asymptotics

Recall that Hardy and Ramanujan tell us that as n goes to infinity, $p(n)$ is asymptotic to $\frac{1}{4n\sqrt{3}} exp(C\sqrt{n})$ where $C = \sqrt{2/3}\pi = 2.565099661\ldots$, and hence $\log p(n)/\sqrt{n}$ converges to C. By looking at the sequence $\log f(n)/\sqrt{n}$ for $1 \leq n \leq 508$, it seems that this too converges to a limit, that appears to be a bit larger than 1.517 (but of course way less than $2.565099661\ldots$). Let's call that constant the *Wilf constant*.

The numerical evidence is here: `http://www.math.rutgers.edu/~zeilberg/tokhniot/oDMP4` .

Let me conclude with two challenges.

• Prove that the Wilf constant exists.

• Determine the exact value of the Wilf constant (if it exists) in terms of π or other famous constants. Failing this, find non-trivial rigorous lower and upper bounds.

Added Feb. 1, 2012: According to Daniel Kane [so far private communication, hopefully he would write it up soon] the *true* asymptotics of $\log f(n)$ is $(6n)^{1/3} log(n)/3$.

Added Feb. 4, 2012: According to Vaclav Kotesovec, this does not agree with the numerics up to 518 terms [but of course, with asymptotics, one may never know for sure]. See

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/dmpKotesovec.gif` .

References

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