Proof of a Conjecture of Philippe di Francesco and Paul Zinn-Justin related to the qKZ equations and to Dave Robbins' Two Favorite Combinatorial Objects

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For any Laurent polynomial $P(z_1, \dots, z_n)$ in the variables $(z_1, \dots, z_n)$, recall that the *anti-symmetrizer*, $AS(P)$ is defined by:

$$AS(P)(z_1, \dots, z_n) = \sum_{\pi \in S_n} (-1)^{inv(\pi)} P(z_{\pi(1)}, \dots, z_{\pi(n)}) \quad ,$$

where S_n is the group of permutations on $\{1, \dots, n\}$ and $inv(\pi)$ is the number of inversions of π (the number of pairs $1 \leq i < j \leq n$ such that $\pi(i) > \pi(j)$).

Also define the *negative-part* of P, $N(P)$, to be the polynomial in $1/z_1, \dots, 1/z_n$ obtained by removing all monomials that have at least one positive exponent.

For example, $AS(5+z_1^{-1}+z_1^3 z_2^{-2}) = z_1^{-1}+z_1^3 z_2^{-2} - z_2^{-1} - z_2^3 z_1^{-2}$, and $N(5+z_1^{-1}+z_1^3 z_2^{-2}) = 5+z_1^{-1}$.

Let

$$F_n(z_1, \dots, z_n) = \prod_{i=1}^{n} z_i^{-2i+2} \prod_{1 \leq i \leq j \leq n} (1 - z_i z_j) \prod_{1 \leq i < j \leq n} (1 + z_i z_j + T z_j) \quad ,$$

$$B_n(z_1, \dots, z_n) = \prod_{1 \leq i < j \leq n} (z_j^{-1} - z_i^{-1})(T + z_i^{-1} + z_j^{-1}) \quad .$$

In [1] (Eq. (4.4)), the following conjecture was made:

$$N\left[AS\left[F_n(z_1, \dots, z_n)\right]\right] = B_n(z_1, \dots, z_n) \quad . \qquad (diFJZ)$$

Let $G_n = AS(F_n)$. By breaking-up the summation over S_n into those π for which $\pi(n) = i$ $(i = 1, \dots, n)$, G_n is seen to satisfy the recurrence

$$G_n(z_1, \dots, z_n) = \sum_{j=1}^{n} (-1)^{n-j} \left(z_j^{1-2n}(1 - z_j^2) \prod_{i=1, i \neq j}^{n} \left[(1 + T z_j + L z_i z_j)(1 - z_i z_j)\right] \right) \cdot$$

$$G_{n-1}(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n) \quad .$$

Let $H_n = N(G_n)$ (our object of desire). Applying the N-operation to both sides, and noting that since the factor in front of $G_{n-1}(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n)$ only has *positive* exponents in $(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n)$,

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we have

$$H_n(z_1,\dots,z_n) = N\{\sum_{j=1}^{n}(-1)^{n-j} z_j^{1-2n}(1-z_j^2)\prod_{i=1,i\neq j}^{n}[(1+Tz_j+Lz_iz_j)(1-z_iz_j)]\cdot$$
$$H_{n-1}(z_1,\dots,z_{j-1},z_{j+1},\dots,z_n)\} \quad .$$

By the inductive hypotheis $H_{n-1}=B_{n-1}$ so we have to prove that

$$B_n(z_1,\dots,z_n) = N\{\sum_{j=1}^{n}(-1)^{n-j} z_j^{1-2n}(1-z_j^2)\prod_{i=1,i\neq j}^{n}[(1+Tz_j+Lz_iz_j)(1-z_iz_j)]\cdot$$
$$B_{n-1}(z_1,\dots,z_{j-1},z_{j+1},\dots,z_n)\} \quad .$$

Now comes a:

Nice Surprise:

$$\sum_{j=1}^{n}\left((-1)^{n-j} z_j^{1-2n}(1-z_j^2)\prod_{i=1,i\neq j}^{n}[(1+Tz_j+Lz_iz_j)(1-z_iz_j)]\right)\cdot B_{n-1}(z_1,\dots,z_{j-1},z_{j+1},\dots,z_n) =$$
$$B_n(z_1,\dots,z_n)(1-z_1^2z_2^2\cdots z_n^2) \quad . \qquad (NiceSurprise)$$

By the *pigeonhole principle*, each and every monomial feauting in $B_n(z_1,\dots,z_n)z_1^2z_2^2\cdots z_n^2$ has at least one positive exponent, hence

$$N(B_n(z_1,\dots,z_n)(1-z_1^2z_2^2\cdots z_n^2)) = B_n(z_1,\dots,z_n) \quad ,$$

and $(diFJZ)$ would follow. It remains to prove the Nice Surprise. By dividing both sides of $(NiceSurprise)$ by B_n, and rearranging terms, we see that it is equivalent to the following Intriguing Identity:

$$\sum_{j=1}^{n}\frac{(2z_j+T)\prod_{i=1}^{n}[(1+z_i(T+z_j))(1-z_iz_j)]}{(1+z_jT+z_j^2)\prod_{i=1,i\neq j}^{n}(z_j-z_i)\prod_{i=1}^{n}(z_j+z_i+T)} = (-1)^{n-1}(1-z_1^2\cdots z_n^2) \quad .$$
$$(IntriguingIdentity)$$

Let $\alpha=\alpha(T)$ and $\beta=\beta(T)$ be the two roots of $1+zT+z^2=0$. Note that $\alpha+\beta=-T$ and $\alpha\beta=1$.

Now it is obvious whom to call for help! Dear old Count Joe!, a.k.a. as Joseph-Louis Lagrange, *Comte de l'Empire*, whose humble (and a posteriori trivial) *Lagrange Interpolation Formula* is at least as useful as Euler-Lagrange and Lagrange multipliers and sum-of-four-squares combined! To wit, for any polynomial $P(z)$ of degree $\leq N-1$ and *any* numbers $w_1,\dots,w_N$, we have

$$P(z) = \sum_{j=1}^{N}\frac{P(w_j)\prod_{i=1,i\neq j}^{N}(z-w_i)}{\prod_{i=1,i\neq j}^{N}(w_j-w_i)} \quad . \qquad (LI)$$

(Proof: both sides are equal at the N numbers $z = w_1, \dots, w_N$, hence they *always* equal (being polynomials of degree $\leq N-1$).

In fact, we will only need the corollary obtained by comparing the coefficient of z^{N-1} on both sides

$$Coeff \quad of \quad z^{N-1} \quad of P(z) = \sum_{j=1}^{N} \frac{P(w_j)}{\prod_{i=1, i\neq j}^{n}(w_j - w_i)} \quad . \qquad (LI')$$

Now Lagrange-Interpolate

$$P(z) := (2z+T)\prod_{i=1}^{n}[(1+z_i(T+z))(1-z_i z)] \quad ,$$

with respect to the $2n+2$ numbers

$$\{z_1, \dots, z_n, -z_1 - T, \dots, -z_n - T, \alpha, \beta\} \quad ,$$

in order to get

$$2(-1)^n z_1^2 \cdots z_n^2 = \sum_{j=1}^{n} \frac{P(z_j)}{(1+z_jT+z_j^2)\prod_{i=1,i\neq j}^{n}(z_j - z_i)\prod_{i=1}^{n}(z_j + z_i + T)}$$

$$+\sum_{j=1}^{n} \frac{P(-T-z_j)}{(1+z_jT+z_j^2)\prod_{i=1,i\neq j}^{n}(-z_j - T - z_i)\prod_{i=1}^{n}(-z_j - T + z_i + T)}$$

$$+\frac{P(\alpha)}{(\alpha-\beta)\prod_{i=1}^{n}(\alpha - z_i)\prod_{i=1}^{n}(\alpha + z_i + T)}$$

$$+\frac{P(\beta)}{(\beta-\alpha)\prod_{i=1}^{n}(\beta - z_i)\prod_{i=1}^{n}(\beta + z_i + T)} \quad .$$

Since $P(-T-z_j) = -P(z_j)$ (check!) and using $\alpha\beta = 1$ and $\alpha + \beta = -T$ to simplify, we get that $(IntriguingIdenity)$ is indeed true. $\Box$.

Reference

1. P. di Francesco and P. Zinn-Justin, *Quantum Knizhnik-Zamolodchikov equation, Totally Symmetric Self-Complementary Plane Partitions and Alternating Sign Matrices*, `http://www.arxiv.org/abs/math-ph/0703015` .