

# All the Moments of the Vertex Degrees of Randomly Generated Graphs

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Consider the set of all labelled graphs on  $n$  vertices and  $m$  edges. By the handshaking lemma the average degree of *every* graph is  $2m/n$ . Higher moments do depend on the graph, but one can easily compute the average of the average of  $\binom{\text{deg}(v)}{k}$ , for any  $k$ . Let  $N = n(n-1)/2$ . The answer is  $\binom{n-1}{k} \binom{N-k}{m-k} / \binom{N}{m}$ , which equals the rational function (in  $n$  and  $m$ )  $\binom{n-1}{k} (m-k+1)_k / (N-k+1)_k$ , where as usual  $(a)_k := a(a+1) \cdots (a+k-1)$ . Indeed, for any particular vertex  $v$ , the sum of  $\binom{\text{deg}(v)}{k}$  over all graphs with  $n$  vertices and  $m$  edges counts such graphs with a  $k$ -subset of  $v$ 's neighbors marked. Clearly the number of these marked graphs is  $\binom{n-1}{k} \binom{N-k}{m-k}$ , (the first term for choosing the marked neighbors and the second term for choosing the remaining edges. From this it is easy to compute any desired moment.

The special case  $k = 2$  yields a one-line proof of the result in **Ivan Gutman and Peter Paule, Univ. Beograd. Publ. Elek. Fak. 13 (2002), 30-35.** [<http://mathematika.etf.bg.ac.yu/publikacije/>].

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