All the Moments of the Vertex Degrees of Randomly Generated Graphs

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Consider the set of all labelled graphs on $n$ vertices and $m$ edges. By the handshaking lemma the average degree of every graph is $2m/n$. Higher moments do depend on the graph, but one can easily compute the average of the average of $\binom{\deg(v)}{k}$, for any $k$. Let $N = n(n-1)/2$. The answer is $\binom{n-1}{k} \binom{N-k}{m-k}/\binom{N}{m}$, which equals the rational function (in $n$ and $m$) $\binom{n-1}{k} (m-k+1)/ (N-k+1)$, where as usual $(a)_k := a(a+1) \cdots (a+k-1)$. Indeed, for any particular vertex $v$, the sum of $\binom{\deg(v)}{k}$ over all graphs with $n$ vertices and $m$ edges counts such graphs with a $k$-subset of $v$’s neighbors marked. Clearly the number of these marked graphs is $\binom{n-1}{k} \binom{N-k}{m-k}$, the first term for choosing the marked neighbors and the second term for choosing the remaining edges. From this it is easy to compute any desired moment.
