Counting Standard Young Tableaux With Restricted Runs

Manuel KAUERS and Doron ZEILBERGER

Dedicated to super-enumerators Ian Goulden and David Jackson[1]

Abstract. The number of standard Young tableaux whose shape is a k by n rectangle is famously $\frac{(nk)!\,0!\,1!\cdots(k-1)!}{(n+k-1)!(n+k-2)!\cdots n!}$, implying that for each specific k, that sequence satisfies a linear recurrence equation with polynomial coefficients of the **first** order. But what about counting standard Young tableaux where certain "run lengths" are forbidden? Then things seem to get much more complicated. In this tribute to the legendary enumerative pair *Goulden & Jackson* we investigate these intriguing sequences, and conjecture that if the number of rows is larger than two, then these sequences are generally **not** P-recursive. On the positive side, we conjecture that these sequences have 'nice' asymptotic behavior. We pledge donations to the OEIS in honor of the first solvers of these conjectures.

Preface

Some combinatorial families are easy to count, for example the number of subsests of an n-element set, that can be computed in *logarithmic* time (in base 2). Also easy is counting the number of permutations, that can be computed in linear-time. Then you have the really hard ones, for example the number of $n \times n$ Latin squares and the number of self-avoiding walks of length n, for which we will probably never know the exact value of the 1000-th term.

Both the number of subsets of an n-element set, 2^n, and the number of permutations, $n!$, as well as the famous Catalan numbers $(2n)!/(n!(n+1)!)$ (OEIS sequence $A108$) are said to have a **closed-form** formula. They satisfy a *first-order* linear recurrence equation with polynomial coefficients

$$a(n+1) - 2a(n) = 0 \quad , \quad a(n+1) - na(n) = 0 \quad , \quad (n+2)a(n+1) - 2(2n+1)a(n) = 0 \quad .$$

(Note that the first equation is even better, it is *constant coefficients*).

Many natural families satisfy the next-best thing to being closed-form, they satisfy a **linear-recurrence equation** with **polynomial** coefficients, but not necessarily of first order. Such sequences, called P-recursive, or *holonomic* (see [KP]), satisfy an equation of the form

$$\sum_{i=0}^{L} p_i(n)a(n+i) \;=\; 0 \quad ,$$

[1] This article was originally submitted to the electronic journal "Algebraic Combinatorics" founded by Goulden and Jackson, following a solicitation for a special issue in honor of Goulden and Jackson. On July 27, 2020 we got an email message from one of the editors-in-chief, Akihiro Munemasa, informing us that, after an initial review, it is "unlikely to meet the standards of depth and originality that the journal is seeking". Consequently this article will remain in the 'Personal journal of Shalosh B. Ekhad and Doron Zeilberger', the homepage of Manuel Kauers, and of course arxiv.org. Let the readers decide about its depth and originality.

for some positive integer L and some **polynomials** in n, $p_0(n), \dots, p_L(n)$.

The most famous such sequences that are **not** closed-forms are the Fibonacci numbers, F_n (OEIS sequence $A45$), the number of involutions of an n-element set (permutations that are equal to their inverse) w_n (OEIS sequence $A85$), and the Motzkin numbers, M_n, the number of words of length n in the alphabet $\{0, -1, 1\}$ that add-up to 0 and all whose partial sums are non-negative (OEIS sequence $A1006$). They satisfy, respectively, the recurrences

$$a(n+2) - a(n+1) - a(n) = 0 \quad , \quad a(n+2) - a(n+1) - (n+1)a(n) = 0 \quad ,$$

$$(n+4)a(n+2) - (2n+5)a(n+1) - (3n+3)a(n) = 0 \quad .$$

Given a natural combinatorial family, parameterized by n, it is very interesting to know whether or not the sequence of integers that enumerates it happens to be P-recursive. This is interesting both conceptually and computationally, since a linear recurrence makes it easy to compute many terms, as well as deriving the asymptotics. Sometimes proving that a given sequence, or family of sequences, is (are) P-recursive is highly non-trivial, see for example [GJR], [GJ], and [Ge2].

In this modest tribute to Ian Goulden and David Jackson we will raise the question whether a certain very *natural* family of combinatorial sequences is P-recursive, and give ample computational evidence that generally they are, *probably*, **not**.

Maple packages

This article is accompanied by the Maple packages `YoungT.txt` and `Tableaux3R.txt`, available from `http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/cyt.html` .

Counting Standard Young Tableaux with Restricted Runs

Recall that a partition of a positive integer m is a weakly-decreasing list of positive integers $\lambda = [\lambda_1, \dots, \lambda_k]$, that add-up to m, also called a *shape*, and a *standard Young tableau* of shape λ is a left-justified array of k rows with λ_i boxes in the i-th row, where the integers $\{1, \dots, m\}$ are filled in the boxes so that both rows and columns are increasing. For example, here is a standard Young tableau of shape $[3, 3, 2]$

$$\begin{matrix} 1 & 3 & 4 \\ 2 & 5 & 7 \\ 6 & 8 & \end{matrix} \quad ,$$

and here is one of shape $[5, 5, 4]$

$$\begin{matrix} 1 & 3 & 4 & 6 & 7 \\ 2 & 5 & 8 & 9 & 11 \\ 10 & 12 & 13 & 14 & \end{matrix} \quad .$$

The number of standard Young tableaux of shape $[\lambda_1, \ldots, \lambda_k]$ is famously given by the **Young-Frobenius** formula (equivalent to the *hook-length formula*)

$$\frac{(\lambda_1 + \ldots + \lambda_k)!}{(\lambda_1 + k - 1)! \cdots \lambda_k!} \cdot \prod_{1 < i < j \leq k} (\lambda_i - \lambda_j + j - i) \quad .$$

In particular, setting $\lambda_1 = \ldots = \lambda_k = n$, we get that the number of standard Young tableaux whose shape is a k by n rectangle is given by

$$\frac{(nk)!\, 0!\, 1! \cdots (k-1)!}{(n+k-1)!(n+k-2)! \cdots n!} \quad .$$

It follows that for each **fixed** k, this sequence, in n, is P-recursive, in fact it even satisfies a **first-order** linear recurrence.

Let's define a **run** in a standard Young tableau to be a **maximal** string of **consecutive integers**. For example, in the following tableau of shape $[5, 5, 5]$

$$\begin{matrix} 1 & 3 & 4 & 6 & 7 \\ 2 & 5 & 8 & 9 & 11 \\ 10 & 12 & 13 & 14 & 15 \end{matrix} \quad ,$$

we have

- First row: one run of length 1, (1), and two runs of length 2 (34 and 67) ;
- Second row: three runs of length 1, (2, 5 and 11) and one run of length 2 (89) ;
- Third row: one run of length 1, (10) and one run of length 4 $(12, 13, 14, 15)$.

We are interested in the following general question. Fix $k > 1$. Given *arbitrary* finite sets of positive integers $A_1, \ldots, A_k$, (or infinite arithmetical progressions), compute the following integer sequence, let's call it

$$G_{A_1,\ldots,A_k}(n) \quad ,$$

defined as the number of standard Young tableaux of shape $(n, \ldots, n)$ such that in each row i , $1 \leq i \leq k$, none of the runs belongs to A_i. Of course if all the A_i's are the empty set, we are back to counting unrestricted standard Young tableaux, for which there is a nice closed-form formula, and of course it is P-recursive.

The case of **two** rows can be shown [EZ] to always give P-recursive sequences (in fact even something stronger is true: the generating functions are algebraic formal power series). This is the case since $2 \times n$ standard Young tableaux are in easy bijection with *Dyck paths* of semi-length n.

More generally, as is well-known, and immediate to see, standard Young tableaux are in easy bijection with **lattice paths**. A k-rowed standard Young tableau of shape $[\lambda_1, \ldots, \lambda_k]$ corresponds to a lattice path in the k-dimensional hyper-cubical lattice, from the origin $[0, \ldots, 0]$ to the point $[\lambda_1, \ldots, \lambda_k]$, with **unit positive steps** $\mathbf{e}_i := [0^{i-1}, 1, 0^{k-i}]$, that **always** stay in the region $x_1 \geq x_2 \geq \ldots \geq x_k$. Given a standard Young tableau, the corresponding path is obtained by executing the step $\mathbf{e}_i$ at the m-th step, if m is located at the i-th row. So $G_{A_1,\ldots,A_k}(n)$ is also the number of k-dimensional lattice paths from the origin to $[n, \ldots, n]$, always staying in $x_1 \geq \ldots x_k \geq 0$, such that the walker never has a run-length parallel to the i-th axis that belongs to the set A_i.

The same question makes sense for general walks, not necessarily those confined to $x_1 \geq x_2 \geq \ldots \geq x_k$. It turns out that for this analogous question the sequences are **always** P-recursive, as we will now show.

Counting Lattice Walks with Restricted Runs

In order to motivate the general case, let's first give yet another proof, a bit more complicated than the usual one, of the very easy fact that the generating function for the number of **all** walks, without restrictions, is given by the generating function

$$\frac{1}{1 - x_1 - \ldots - x_k} \quad .$$

Every walk corresponds to a **word** in the alphabet $\{1, \ldots k\}$, indicating which $\mathbf{e}_i$ it went through. For example, the walk

$$[0,0,0] \rightarrow [1,0,0] \rightarrow [1,0,1] \rightarrow [1,1,1] \rightarrow [1,2,1] \rightarrow [1,2,2] \rightarrow [1,3,2]$$

corresponds to the word

$$132232 \quad .$$

Given a word in $\{1, \ldots, k\}$, we can write it in **frequency notation** $b_1^{r_1} \ldots b_l^{r_l}$, where $b_{j+1} \neq b_j$, and $r_j \geq 1$. for example, the above word 132232 is abbreviated $1^1 3^1 2^2 3^1 2^1$, and the word 1113322211 is written $1^3 3^2 2^3 1^2$.

Let $F_i = F_i(x_1, \ldots, x_k)$ be the **weight-enumerator** of all words that end with the letter i. Then obviously, for $i = 1, \ldots, k$

$$F_i \; = \; \frac{x_i}{1 - x_i} \left(1 + \sum_{\substack{1 \leq j \leq k \\ j \neq i}} F_j \right) \quad .$$

This is a system of k linear equations with k unknowns $F_1, \ldots, F_k$, whose solution is easily seen to be given explicitly by

$$F_i = \frac{x_i}{1 - x_1 - \ldots - x_k} \quad .$$

Finally the full generating function, F, is gotten by adding the weight of the **empty** word, 1, to the sum of the F_i's, getting

$$F = 1 + \sum_{i=1}^{k} F_i \quad ,$$

that implies the deep theorem

$$F = \frac{1}{1 - x_1 - \ldots - x_k} \quad .$$

Note, in particular that the F_i, (and F) are **rational functions** of the variables $x_1, \ldots, x_k$.

To handle the restricted case, to find the weight-enumerator of all words in $1^{\lambda_1} \ldots k^{\lambda_k}$ such that when written in frequency notation $b_1^{r_1} \ldots b_m^{r_m}$ we have that if $b_\alpha = i$ then $r_\alpha \notin A_i$ (i.e. runs in the $\mathbf{e}_i$ direction can't be of a length that belongs to A_i), we have the modified system:

$$F_i \ = \left(\frac{x_i}{1 - x_i} - \sum_{\beta \in A_i} {x_i}^{\beta} \right) \left(1 + \sum_{\substack{1 \leq j \leq k \\ j \neq i}} F_j \right) \quad .$$

This is a system of k linear equations in the k unknowns $F_1, \ldots, F_k$, with coefficients that are **rational functions** in $x_1, \ldots, x_k$. Hence, by Cramer's rule, the F_i are all rational functions of $x_1, \ldots, x_k$, and hence so is $F = 1 + \sum_{i=1}^{k} F_i$.

Our sequence of interest is the sequence of coefficients of the **diagonal** of this rational function. Since the diagonal of any formal power series that is a rational function is D-finite (see [Ge1][Z1][L][Z2]), it follows that the sequence itself is P-recursive.

Back to Tableaux

We strongly doubt that the multi-variable generating functions for restricted Young tableaux (alias restricted walks confined to $x_1 \geq \ldots \geq x_k$) are rational. In order to explore these sequences, we need to generate as many terms as possible. Here is how to do it. Let us fix $R_1, \ldots, R_k$ and denote by $g(\lambda_1, \ldots, \lambda_k)$ the number of standard Young tableaux of shape $[\lambda_1, \ldots, \lambda_k]$ with no runs in the i-th row that belong to R_i, or equivalently the number of walks in $x_1 \geq \ldots \geq x_k \geq 0$, from the origin to the point $[\lambda_1, \ldots, \lambda_k]$ with no run-length in the x_i-direction that belongs to R_i (for $i = 1, \ldots, k$).

In order to compute $g(\lambda_1, \ldots, \lambda_k)$, we need the more refined quantities $(1 \leq i \leq k)$ $g^{(i)}(\lambda_1, \ldots, \lambda_k)$, that enumerate those walks that end with a step in the x_i-direction.

We have the **dynamic programming** recurrences $(1 \leq i \leq k)$

$$g^{(i)}(\lambda_1, \ldots, \lambda_k) \ = \sum_{\substack{1 \leq j \leq k \\ j \neq i}} \ \sum_{\substack{1 \leq r \leq \lambda_i \\ r \notin A_i}} g^{(j)}(\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - r, \lambda_{i+1}, \ldots, \lambda_k) \quad ,$$

with the obvious initial conditions, and the **boundary conditions**

$$g^{(i)}(\lambda_1, \ldots, \lambda_k) \ = \ 0 \quad ,$$

whenever $\lambda_1 < \lambda_2$ or $\lambda_2 < \lambda_3$, ..., or $\lambda_{k-1} < \lambda_k$, or $\lambda_k < 0$. Finally

$$g(\lambda_1, \ldots, \lambda_k) = \sum_{i=1}^{k} g^{(i)}(\lambda_1, \ldots, \lambda_k) \quad .$$

This is all implemented in the Maple package `YoungT.txt` mentioned above.

Two Case Studies

In spite of the fact that we were unable to think of a good reason why these sequences should be P-recursive, we still hoped that they would be for a non-obvious reason. We focused on two special cases to generate as many terms as we could.

• $G(n)$, the number of standard Young tableaux of shape $[n,n,n]$ where each run, in each of the three rows, must have length at least 2. This is the case $A_1 = A_2 = A_3 = \{1\}$ in the above notation.

• $H(n)$, the number of standard Young tableaux of shape $[n,n,n]$ where all the run-lengths, in each row are always odd. This is the case $A_1 = A_2 = A_3 = \{2r+2 : r \geq 0\}$ in the above notation.

Regarding $G(n)$, using the Maple package

`http://www.math.rutgers.edu/~zeilberg/tokhniot/Tableaux3R.txt`

we got that the sequence starts with (starting at $n = 1$)

$$0, 1, 1, 5, 15, 69, 304, 1518, 7807, 42314, 236621, 1364570, 8062975, 48680547, 299388670, 1871463427, \dots \quad .$$

The first 200 terms may be viewed here:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oTableaux3R1.txt` .

The first 996 terms are available here:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/CYT/GseqList.txt` .

Regarding $H(n)$, we got that the sequence starts with (starting at $n = 1$)

$$1, 2, 9, 46, 306, 2252, 18308, 158872, 1454570, 13888112, 137277741, 1396638636, 14561307281, 155040525128, \dots \quad .$$

The first 200 terms can be viewed here:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oTableaux3R2.txt` .

The first 965 terms are available here:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/CYT/HseqList.txt` .

Even that many terms were not enough to guess a linear recurrence with polynomial coefficients, so if such a recurrence exists, it would be extremely complicated. But we can do better! The existence of a non-zero linear recurrence of a given order and degree boils down to the existence of a non-zero solution to a certain system of linear equations with integer coefficients. If a non-trivial solution exists, then doing everything modulo any prime would also have a solution. Conversely, if there is no solution modulo that prime, there is no solution at all. Now we can generate many more terms, and using the prime $p = 45007$ we (or rather our computer) generated 5000 terms, and even these did not suffice. In other words if there exists such a recurrence of order $\leq K$ and degree, in n, of the coefficients of degree $\leq K$, then $(K+1)^2 + 5 \geq 5000$, i.e. $K \geq 70$.

This leads us to make the following conjectures. One of us (DZ) is pledging a donation of 200 US dollars to the On-Line Encyclopedia of Integer Sequences in honor of the first prover, for each of the following four conjectures.

Conjecture 1a: The sequence $G(n)$ is **not** P-recursive.

Conjecture 1b: The sequence $H(n)$ is **not** P-recursive.

Surprisingly, the asymptotics seems to be very nice. Using the nearly 1000 terms in these sequences we are safe in making the following conjectures.

Conjecture 2a: There exists a constant C_1 (if possible, find it!) such that

$$G(n) \asymp C_1 \frac{8^n}{n^4} \quad .$$

We estimate C_1 to be close to 0.521286 .

Conjecture 2b: There exists a constant C_2 (if possible, find it!) such that

$$H(n) \asymp C_2 \frac{(7+5\sqrt{2})^n}{n^4} \quad .$$

We estimate C_2 to be close to 0.63892.

This raises the more general question about these sequences. Is the asymptotics always of the form $C\mu^n n^\theta$ with μ an algebraic number, and θ a rational number?

References

[EZ] Shalosh B. Ekhad and Doron Zeilberger, *Automatic counting of restricted Dyck paths via (numeric and symbolic) Dynamic Programming*, The Personal Journal of Shalosh B. Ekhad and Doron Zeilberger, June 3, 2020.
`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/dyck.html`.
Also available from: `https://arxiv.org/abs/2006.01961` .

[Ge1] Ira Gessel, *Two theorems of rational power series*, Utilitas Math. **19** (1981), 247-254.

[Ge2] Ira Gessel, *Counting Latin rectangles*, Bull. Amer. Math. Soc. (N.S.) **16** (1987), 79-82.

[GJR] Ian P. Goulden, David M. Jackson, and James W. Reilly, *The Hammond series of a symmetric function and its application to P-recursiveness*, SIAM J. Algebraic Discrete Methods **4** (1983), 179-193.

[GJ] Ian P. Goulden and David M. Jackson, *Labelled graphs with small vertex degrees and P-recursiveness*, SIAM J. Algebraic Discrete Methods **7** (1986), 60-66.

[KP] Manuel Kauers and Peter Paule, *"The Concrete Tetrahedron"*, Springer, 2011.

[L] Leonard Lipshitz, *The diagonal of a D-finite power series is D-finite*, J. Algebra **113** (1988), 373-378.

[Z1] Doron Zeilberger, *Sister Celine's technique and its generalizations*, J. Math. Anal. Appl. **85** (1982), 114-145.

[Z2] Doron Zeilberger, *A holonomic systems approach to special functions identities*, J. Comput. Appl. Math. **32** (1990), 321-368.

Manuel Kauers, Institute for Algebra, J. Kepler University Linz, Austria E-mail `manuel dot kauers at jku dot at`

Doron Zeilberger, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.
Email: `DoronZeil at gmail dot com` .

First Written: June 20, 2020. This version: July 27, 2020.

Exclusively published in the Personal Journal of Shalosh B. Ekhad and Doron Zeilberger, Manuel Kauers' website, and arxiv.org .